

p -groups having a unique proper non-trivial characteristic subgroup

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ABSTRACT. We consider the structure of finite p -groups G having precisely three characteristic subgroups, namely 1 , $\Phi(G)$ and G . The structure of G varies markedly depending on whether G has exponent p or p^2 , and, in both cases, the study of such groups raises deep problems in representation theory. We present classification theorems for 3- and 4-generator groups, and we also study the existence of such r -generator groups with exponent p^2 for various values of r . The automorphism group induced on the Frattini quotient is, in various cases, related to a maximal linear group in Aschbacher's classification scheme.

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1. INTRODUCTION

Taunt [Tau55] considered groups having precisely three characteristic subgroups. As such groups have a **unique proper non-trivial characteristic subgroup**, he called these UCS-groups. He gave necessary, but not sufficient, conditions for the direct power of a UCS-group to be a UCS-group. Taunt discussed solvable UCS-groups in [Tau55], and promised a forthcoming paper describing the structure of UCS p -groups. However, his article on UCS p -groups was never written. The present paper is devoted to the study of these groups.

In our experience UCS p -groups are rather elusive, and it is unlikely that a general classification could be given. However, the study of these groups does lead to the exploration of very interesting problems in representation theory, and some interesting, sometimes even surprising, theorems can be proved.

The main results of this paper can be summarized as follows.

Theorem 1. *Let G be a non-abelian UCS p -group where $|G/\Phi(G)| = p^r$.*

- (a) *If $r = 2$, then G belongs to a unique isomorphism class.*
- (b) *If $r = 3$, then G belongs to a unique isomorphism class if $p = 2$, and to one of two distinct isomorphism classes if $p > 2$.*
- (c) *If $r = 4$ and either $p = 2$ or G has exponent p , then G belongs to one of eight distinct isomorphism classes.*
- (d) *Suppose that $r = 4$ and G has exponent p^2 . Then $p \neq 5$. Further, if $p \equiv \pm 1 \pmod{5}$ then G belongs to a unique isomorphism class; while if $p \equiv \pm 2 \pmod{5}$ then G belongs to one of two distinct isomorphism classes.*
- (e) *Let p be an odd prime, and let k be a positive integer. Then there exist non-abelian exponent- p^2 UCS-groups*
 - (i) *of order p^{6k} for all p and k ;*
 - (ii) *of order p^{10} if and only if $p^5 \equiv 1 \pmod{11}$;*
 - (iii) *of order p^{14k} for all p and k .*

Parts (a) and (b) of Theorem 1 follow from Theorem 8, while part (c) follows from Theorems 10 and 11. Part (d) is verified in Theorem 20 and the proof of (e) can be found at the end of Section 7.

As UCS p -groups have precisely three characteristic subgroups, they must have exponent p or p^2 . Abelian UCS p -groups are all of the form $(C_{p^2})^r$ where p is a prime. Non-abelian UCS p -groups with exponent p are quite common, and so we could investigate

them only for small generator number (at most 4). However, it seems that non-abelian UCS p -groups with exponent p^2 are less common, and we could even prove that for certain choices of the pair (p, r) in Theorem 1 they do not exist. For instance Theorem 1(d) implies that there is no non-abelian UCS 5-group with 4 generators and exponent 25. In both cases, the study of such groups leads to challenging problems in representation theory. A well-written introduction to p -groups of Frattini length 2 can be found in [GQ06].

Let p be an odd prime, and let G be a non-abelian r -generator UCS p -group with exponent p . Then G has the form H/N where H is the r -generator, free group with exponent p and nilpotency class 2, and N is a proper subgroup of H' . The groups H/H' and H' are elementary abelian, and so they can be considered as vector spaces over \mathbb{F}_p . Moreover, the group $\mathrm{GL}_r(p)$ acts on H/H' in the natural action, and on H' in the exterior square action. It is proved in Theorem 6 that G is UCS if and only if the (setwise) stabilizer $K := \mathrm{GL}_r(p)_N$ is irreducible on both H/H' and H'/N . Further, an irreducible subgroup $K \leq \mathrm{GL}_r(p)$ and an irreducible K -factor module H'/N lead to a UCS p -group with exponent p (Theorem 7). Thus the investigation of the class of UCS p -groups with exponent p is reduced to the study of irreducible subgroups K of $\mathrm{GL}_r(p)$ and the maximal K -submodules of the exterior square $\Lambda^2(\mathbb{F}_p)^r$.

The study of UCS 2-groups poses different problems than the odd case. In this paper we concentrate on UCS groups of odd order, and prove only a few results about UCS 2-groups.

A non-abelian UCS p -group G with *odd* order and exponent p^2 , is a *powerful* p -group (i.e. $G^p \leq G'$). Moreover, G has the form H/N where H is the r -generator free group with p -class 2 and exponent p^2 and N is a subgroup of $\Phi(H)$. As above, $\mathrm{GL}_r(p)$ acts on $H/\Phi(H)$ in the natural action and $\Phi(H) = H^p \oplus H'$ can also be viewed as a $\mathrm{GL}_r(p)$ -module. The commutator subgroup G' is isomorphic to $H'/(H' \cap N)$ and, as G has one proper non-trivial characteristic subgroup, G' must coincide with G^p . On the other hand, as the p -th power map $x \mapsto x^p$ is a homomorphism, the $\mathrm{GL}_r(p)_N$ actions on $H/\Phi(H)$ and on $H'/(H' \cap N)$ must be equivalent. Further $\mathrm{GL}_r(p)_N$ must be irreducible on both $H/\Phi(H)$ and $H'/(H' \cap N)$ (see Theorem 6 for the proof of the last two assertions). Therefore we found an irreducible subgroup K of $\mathrm{GL}_r(p)$ and a maximal K -submodule N in $\Lambda^2(\mathbb{F}_p)^r$ such that $\Lambda^2(\mathbb{F}_p)^r/N$ is equivalent to the natural module of K . Conversely, such a group K and a submodule N lead to a UCS p -group with exponent p^2 (see Theorem 7). Thus UCS p -groups with odd exponent p^2 give rise to irreducible modules that are isomorphic

to a quotient of the exterior square. We call these *exterior self-quotient* modules (or ESQ-modules). We are planning to write a paper devoted to the study of ESQ-modules.

In Sections 4 and 6, UCS p -groups with exponent p and generator number at most 4 are studied. We achieve a complete classification of 2- and 3-generator UCS p -groups and, for odd p , a complete classification of 4-generator UCS p -groups with exponent p . These classifications are made possible in these cases by our knowledge of the $\mathrm{GL}_r(p)$ -module $W = \Lambda^2(\mathbb{F}_p)^r$. If $r = 2$ then W is a 1-dimensional module and our problem is trivial. If $r = 3$, then W is equivalent to the dual of the natural module and the classification of UCS p -groups with exponent p is straightforward also in this case. However, the classification of 3-generator UCS p -groups with exponent p^2 is already non-trivial. The fact that there is, up to isomorphism, precisely one 3-generator UCS p -group with exponent p^2 is a consequence of the fact that the stabilizer in the general linear group $\mathrm{GL}_3(p)$ of a certain 3-dimensional subspace of $(\mathbb{F}_p)^6$ is the special orthogonal group $\mathrm{SO}_3(p)$ (see Lemma 9).

When $r = 4$, the Klein correspondence makes it possible to obtain the necessary information about the $\mathrm{GL}_r(p)$ -module W . In this case $\mathrm{GL}_r(p)$ preserves (up to scalar multiples) a quadratic form on W . A p -group with exponent p corresponds to a subspace N of W as explained above. These observations and the classification of p -groups with order dividing p^7 (see [NOVL04, OVL05]) enables us to classify 4-generator UCS p -groups with exponent p . We found it surprising that N leads to a UCS p -group if and only if the restriction of the quadratic form to N is non-degenerate. The details are presented in Section 6. A brief classification of 4-generator UCS 2-groups is given in Section 5.

Section 7 focusses on the construction of exterior self-quotient (ESQ) modules for dimensions at most 5. As remarked above, this is directly related to the construction of UCS p -groups with odd exponent p^2 . Our results for exponent- p^2 UCS groups are summarized in Theorem 1(d,e), and proved in Section 7. Theorems 18, 19 are concerned with the structure of ESQ-modules in dimensions 4 and 5, and are of independent interest.

2. p -GROUPS WITH PRECISELY 3 CHARACTERISTIC SUBGROUPS

We shall focus henceforth on the structure of a p -group G with precisely three characteristic subgroups. In such a group the Frattini subgroup $\Phi(G)$ is non-trivial, otherwise G is elementary abelian and characteristically simple. Therefore the non-trivial, proper characteristic subgroup of G is $\Phi(G)$.

Let p be a prime number. The *lower p -central series* of a group H (which may not be a p -group) is defined as follows: $\lambda_1^p(H) = H$, and $\lambda_i^p(H) = [\lambda_{i-1}^p(H), H](\lambda_{i-1}^p(H))^p$ for $i \geq 2$. For a finite p -group G , the subgroup $\lambda_i^p(G)$ is denoted by $\lambda_i(G)$. The term $\lambda_2(G)$ coincides with $\Phi(G)$. The *p -class* of a finite p -group G is defined as the smallest integer c such that $\lambda_{c+1}(G) = 1$. Clearly, terms of the lower p -central series of a finite p -group G are characteristic, and G has p -class 1 if and only if G is elementary abelian. This simple observation leads to the following lemma.

Lemma 2. *The p -class of a UCS p -group is precisely 2.*

A UCS p -group can be written as a quotient of a suitable free group which we now describe. Let p be a prime, and let r be a positive integer. Let F_r denote the free group with rank r , and set

$$H_{p,r} = F_r / \lambda_3^p(F_r).$$

The group $H_{p,r}$ is an r -generator free group in the variety of groups with p -class 2. Since $\lambda_3^p(H_{p,r}) = 1$, we have that $H_{p,r}$ is an r -generator nilpotent group whose exponent divides p^2 , and so $H_{p,r}$ is a finite p -group. The quotient $H_{p,r}/\Phi(H_{p,r})$ is elementary abelian with rank r . As $\lambda_3(H_{p,r}) = [\Phi(H_{p,r}), H_{p,r}]\Phi(H_{p,r})^p = 1$, the subgroup $\Phi(H_{p,r})$ is elementary abelian and central. Assume that the elements x_1, \dots, x_r form a minimal generating set for $H_{p,r}$. The Frattini subgroup $\Phi(H_{p,r})$ is minimally generated by the elements x_i^p and $[x_j, x_k]$ with $i, j, k \in \{1, \dots, r\}$ and $j < k$. Thus $\Phi(H_{p,r})$ has rank $r(r-1)/2 + r$. In 2-groups, the subgroup generated by the squares contains the commutator subgroup, so $\Phi(H_{2,r}) = (H_{2,r})^2$. On the other hand, if $p \geq 3$, then $\Phi(H_{p,r}) = (H_{p,r})' \oplus (H_{p,r})^p$. In this case, the commutators $[x_j, x_k]$ with $j < k$ form a minimal generating set for $(H_{p,r})'$, while the p -th powers x_i^p form a minimal generating set for $(H_{p,r})^p$. Thus $(H_{p,r})'$ and $(H_{p,r})^p$ are elementary abelian with ranks $r(r-1)/2$ and r , respectively.

When investigating UCS p -groups G , by the following lemma, we may conveniently assume that G is of the form $H_{p,r}/N$ where $N \leq \Phi(H_{p,r})$. The proof of the following lemma is straightforward.

Lemma 3. *Let p be a prime, let G be an r -generator UCS p -group, and let $H_{p,r}$ be the group above. Suppose that $\{x_1, \dots, x_r\}$ and $\{y_1, \dots, y_r\}$ are minimal generating sets of $H_{p,r}$ and G , respectively. Then the mapping $x_i \mapsto y_i$, for $i \in \{1, \dots, r\}$, can uniquely be extended to an epimorphism $\varphi : H_{p,r} \rightarrow G$. Further, the kernel of φ lies in $\Phi(H_{p,r})$.*

Now we introduce some notation that will be used in the rest of the paper. If L is a group that acts on a vector space V , then L^V denotes the image of L under this action. Hence $L^V \leq \text{GL}(V)$. The stabilizer in L of an object X is denoted by L_X . If G is a p -group, then \overline{G} denotes $G/\Phi(G)$. If G is a UCS p -group, then $\Phi(G)$ and \overline{G} can be considered as \mathbb{F}_p -vector spaces. We shall consider the linear groups $\text{Aut}(G)^{\Phi(G)}$ and $\text{Aut}(G)^{\overline{G}}$.

Set $H = H_{p,r}$. The subgroup $\Phi(H)$ can be viewed as a $\text{GL}(\overline{H})$ -module as follows. Recall that $H = F_r/\lambda_3^p(F_r)$ and $\Phi(H) = \lambda_2^p(F_r)/\lambda_3^p(F_r)$. Let $g \in \text{GL}(F_r/\lambda_2^p(F_r))$ and $x, y \in F_r$. Set $(x^p\lambda_3^p(F_r))g = \hat{x}^p\lambda_3^p(F_r)$ and $([x, y]\lambda_3^p(F_r))g = [\hat{x}, \hat{y}]\lambda_3^p(F_r)$ where \hat{x} and \hat{y} are chosen so that $(x\lambda_2^p(F_r))g = \hat{x}\lambda_2^p(F_r)$ and $(y\lambda_2^p(F_r))g = \hat{y}\lambda_2^p(F_r)$. It is proved in [O'B90, Lemma 2.6] that this rule defines a unique linear transformation on $\Phi(H)$ corresponding to g .

The following lemma describes the structure of the $\text{GL}(\overline{H})$ -module $\Phi(H)$; see for instance the argument on page 26 in [Hig60].

Lemma 4. *Let p be a prime, let r be an integer, and set $H = H_{p,r}$. The subgroup H' is a $\text{GL}(\overline{H})$ -submodule of $\Phi(H)$ and $\Phi(H)/H' \cong \overline{H}$ as $\text{GL}(\overline{H})$ -modules. If $p \geq 3$, then $\Phi(H) = H' \oplus H^p$ is a direct sum of $\text{GL}(\overline{H})$ -modules. In particular, $\overline{H} \cong H^p$ as $\text{GL}(\overline{H})$ -modules if $p \geq 3$.*

Next we give a description of the automorphism group of G following [O'B90, Theorem 2.10]. If G is of the form $H_{p,r}/N$, with some $N \leq \Phi(H_{p,r})$ then the spaces $\overline{H_{p,r}}$ and \overline{G} can naturally be identified, and this fact is exploited in the following lemma.

Lemma 5. *Let p be a prime, let r be an integer, and set $H = H_{p,r}$. Let N be a subgroup of $\Phi(H)$, and set $G = H/N$. Identifying \overline{G} and \overline{H} , we obtain that*

$$\text{Aut}(G)^{\overline{G}} = \text{GL}(\overline{H})_N \quad \text{and} \quad \text{Aut}(G)^{\Phi(G)} = (\text{GL}(\overline{H})_N)^{\Phi(H)/N}.$$

Further, the kernel K of the $\text{Aut}(G)$ -action on \overline{G} is an elementary abelian p -group of order $|\Phi(G)|^r$, and K acts trivially on $\Phi(G)$.

Lemma 5 enables us to characterize UCS p -groups.

Theorem 6. *Let p be a prime, let r be an integer, set $H = H_{p,r}$, and let $G = H/N$ where $N \leq \Phi(H)$. Then the following are equivalent:*

- (a) G is a UCS p -group;
- (b) both $\text{Aut}(G)^{\overline{G}}$ and $\text{Aut}(G)^{\Phi(G)}$ are irreducible;
- (c) both $\text{GL}(\overline{H})_N$ and $(\text{GL}(\overline{H})_N)^{\Phi(H)/N}$ are irreducible.

Further, if p is odd and G is a UCS p -group, then precisely one of the following must hold:

- (i) $N = H'$ and G is abelian;
- (ii) $H^p \leq N$ and G is non-abelian of exponent p ;
- (iii) $N \cap H^p = 1$, \overline{H} and $H'/(N \cap H')$ are equivalent $\text{GL}(\overline{H})_N$ -modules, G is non-abelian of exponent p^2 , and $|G| = p^{2r}$.

Proof. Assertions (b) and (c) are equivalent by Lemma 5. We now prove that (a) and (b) are equivalent. The following two observations show that (a) implies (b). First, inverse images of $\text{Aut}(G)^{\overline{G}}$ -submodules of \overline{G} correspond bijectively to characteristic subgroups of G containing $\Phi(G)$. Second, $\text{Aut}(G)^{\Phi(G)}$ -submodules of $\Phi(G)$ correspond bijectively to characteristic subgroups of G contained in $\Phi(G)$. Assume now that (b) holds and that L is a characteristic subgroup of G . As $L\Phi(G)$ is characteristic, it follows from the observation above that $L\Phi(G)$ equals $\Phi(G)$ or G . In the latter case, $L = G$ as $\Phi(G)$ comprises the set of elements of G that can be omitted from generating sets. In the former case $L\Phi(G) = \Phi(G)$ and $L \leq \Phi(G)$. It follows from (b) that L equals 1 or $\Phi(G)$. Thus (b) implies (a).

We now prove the second statement of the theorem. Suppose that G is a UCS p -group, where p is odd. Suppose additionally that G is abelian. Then $H' \leq N$. Now $H/H' \cong (C_{p^2})^r$ is homocyclic of rank r and exponent p^2 . As in a UCS p -group the subgroups G^p and $\{g \in G \mid g^p = 1\}$ coincide, we obtain that $N = H'$ and (i) holds. Suppose now that G is non-abelian. By Lemma 4, $H'N$ is invariant under $\text{GL}(\overline{H})_N$. By Lemma 5, $\text{Aut}(G)^{\Phi(G)} = (\text{GL}(\overline{H})_N)^{\Phi(H)/N}$, which shows that $H'N/N$ is characteristic in G . Thus $H'N$ equals $\Phi(H)$ or N . As G is non-abelian $H'N = \Phi(H)$ holds. As p is odd, Lemma 4 implies that H^p is invariant under $\text{GL}(\overline{H})$, and so $H^p \cap N$ must be invariant under $\text{GL}(\overline{H})_N$. On the other hand, the first part of Theorem 6 shows that $\text{GL}(\overline{H})_N$ is irreducible on \overline{H} , and so, by Lemma 4, also on H^p . Thus either $N \cap H^p = H^p$ or $N \cap H^p = 1$. In the former case $H^p \leq N$, and case (ii) holds. In the latter case, we shall show below that case (iii) holds.

Suppose now that $H'N = \Phi(H)$, $N \cap H^p = 1$ and p is odd. As H^p is invariant under $\text{GL}(\overline{H})$ (see Lemma 4), NH^p/N must be invariant under $\text{GL}(\overline{H})_N$. Thus Lemma 5 implies that NH^p/N is characteristic in G . Hence $NH^p = \Phi(H)$. As $N \cap H^p = 1$, we obtain the following isomorphisms between $\text{GL}(\overline{H})_N$ -modules:

$$H^p \cong \frac{H^p}{N \cap H^p} \cong \frac{H^p N}{N} = \frac{\Phi(H)}{N} = \frac{H'N}{N} \cong \frac{H'}{N \cap H'}.$$

By Lemma 4, $H^p \cong \overline{H}$, as $\text{GL}(\overline{H})$ -modules, so \overline{H} and $H'/(N \cap H')$ are equivalent $\text{GL}(\overline{H})_N$ -modules. It follows from the above displayed equation that $|N| = |H'| = p^{r(r-1)/2}$, and hence that $|G| = p^{2r}$. Thus case (iii) holds. \square

3. THE EXISTENCE OF UCS p -GROUPS

In this section we study the question whether UCS p -groups exist with given parameters. We find that the existence of UCS p -groups is equivalent to the existence of certain irreducible linear groups.

Let $H = H_{p,r}$ for some prime p and integer r as above. Note that the $\text{GL}(\overline{H})$ -action on H' is equivalent to the exterior square of the natural action. If p is an odd prime and G is an r -generator UCS p -group with exponent p , then G is non-abelian and has the form H/N where $H^p \leq N < \Phi(H)$. Theorem 6 implies that $\text{GL}(\overline{H})_N$ must be irreducible on \overline{H} and also on $\Phi(H)/N$. As $H' \not\leq N$, we obtain that

$$\frac{\Phi(H)}{N} = \frac{H'N}{N} \cong \frac{H'}{H' \cap N}.$$

Denote \overline{G} by V . The argument above shows that an exponent- p UCS p -group G gives rise to an irreducible linear group $K := \text{Aut}(G)^{\overline{G}}$ acting on V , and a maximal K -module M of $\Lambda^2 V$.

The structure of UCS p -groups with exponent p^2 is, by Theorem 6(iii), intimately related to the following (apparently new) concept in representation theory.

Definition. An $\mathbb{F}G$ -module V is called an *exterior self-quotient* module, briefly an *ESQ*-module, if there is an $\mathbb{F}G$ -submodule U of $\Lambda^2 V$ such that $(\Lambda^2 V)/U$ is isomorphic to V . If G acts faithfully on V , we call G an *ESQ-subgroup* of $\text{GL}(V)$, or simply an *ESQ-group*.

By Theorem 6, $\text{Aut}(G)^{\overline{G}}$ is an irreducible ESQ-group when G is a non-abelian UCS p -group of exponent p^2 for odd p . In Section 7 we study exterior self-quotient modules, where it is natural to consider fields other than \mathbb{F}_p . When we consider *necessary* conditions for the existence of UCS p -groups then we usually work over a finite prime field \mathbb{F}_p ; however, for *sufficient* conditions, working over arbitrary (finite) fields is most natural.

Suppose that V is a d -dimensional vector space over a field \mathbb{F}_q where $q = p^k$ for some prime p and integer k . We may consider V as a vector space over the prime field \mathbb{F}_p and also over the larger field \mathbb{F}_q . Thus we may take the exterior squares $\Lambda_{\mathbb{F}_p}^2 V$ and $\Lambda_{\mathbb{F}_q}^2 V$

over \mathbb{F}_p and \mathbb{F}_q , respectively. There is an \mathbb{F}_p -linear map $\varepsilon : \Lambda_{\mathbb{F}_p}^2 V \rightarrow \Lambda_{\mathbb{F}_q}^2 V$ satisfying $\varepsilon(u \wedge v) = u \wedge v$ for all $u, v \in V$.

Theorem 7. *Let p be an odd prime, let r and s be integers.*

(a) *The following two assertions are equivalent.*

- (a1) *There exists a UCS p -group G with exponent p such that $|\overline{G}| = p^r$ and $|\Phi(G)| = p^s$.*
- (a2) *There exists an irreducible linear group K acting on a vector space V over \mathbb{F}_{p^k} , for some k , such that $\dim V = r/k$ and $\Lambda_{\mathbb{F}_{p^k}}^2 V$ has a maximal $\mathbb{F}_{p^k} K$ -submodule with codimension s/k .*

(b) *The following two assertions are equivalent.*

- (b1) *There exists a UCS p -group G with exponent p^2 such that $|\overline{G}| = p^r$.*
- (b2) *There exists an irreducible ESQ-module V over a field \mathbb{F}_{p^k} such that $\dim V = r/k$, and V can not be written over any proper subfields of \mathbb{F}_{p^k} .*

Proof. (a) Assume (a1) is true, and G is an exponent- p UCS-group. Then by Theorem 6 $\text{Aut}(G)^{\overline{G}}$ is an irreducible linear group, so (a2) is true with $k = 1$. Assume now that (a2) holds and set $q = p^k$. Let U be a maximal $\mathbb{F}_q K$ -submodule of $\Lambda_{\mathbb{F}_q}^2 V$ of codimension s/k . Let ε denote the epimorphism $\Lambda_{\mathbb{F}_p}^2 V \rightarrow \Lambda_{\mathbb{F}_q}^2 V$. Let Z denote the group of the non-zero scalar transformations $\{\lambda I \mid \lambda \in \mathbb{F}_q^\times\}$ of V . Then Z commutes with K and so one can form the subgroup ZK . Since K is irreducible on V over \mathbb{F}_q and the action of \mathbb{F}_q^\times on V is realized by Z , we find that ZK is irreducible on V over \mathbb{F}_p . We claim that ZK is irreducible on $\Lambda_{\mathbb{F}_q}^2 V/U$ over \mathbb{F}_p . Note that the element $\lambda I \in Z$ induces the scalar transformation $\lambda^2 I$ on $\Lambda_{\mathbb{F}_q}^2 V/U$. Since these transformations generate the \mathbb{F}_p -algebra $\{\lambda I \mid \lambda \in \mathbb{F}_q\}$ of all scalar transformations of $\Lambda_{\mathbb{F}_q}^2 V/U$, we obtain that an $\mathbb{F}_p ZK$ -submodule of $\Lambda_{\mathbb{F}_q}^2 V/U$ is also an $\mathbb{F}_q ZK$ -submodule. Since K is irreducible on $\Lambda_{\mathbb{F}_q}^2 V/U$ over \mathbb{F}_q , we obtain that ZK is irreducible on $\Lambda_{\mathbb{F}_q}^2 V/U$ over \mathbb{F}_p .

Obviously, ε is a ZK -homomorphism. If $\hat{U} = \varepsilon^{-1}(U)$, then clearly $\Lambda_{\mathbb{F}_q}^2 V/U \cong \Lambda_{\mathbb{F}_p}^2 V/\hat{U}$. Hence \hat{U} is a maximal ZK -submodule in $\Lambda_{\mathbb{F}_p}^2 V$. By this argument, we may, and shall, henceforth assume that (a2) is true with $k = 1$ and will write Λ^2 for $\Lambda_{\mathbb{F}_p}^2$.

Let H denote $H_{p,r}$. As the $\text{GL}(\overline{H})$ -action on H' is equivalent to its action on $\Lambda^2 \overline{H}$, we identify V with \overline{H} , $\Lambda^2 V$ with H' , and \hat{U} with a K -invariant normal subgroup N of index p^s in H' . Set $G = H/(H^p N)$. As $p \geq 3$, Lemma 4 shows that H^p and H' are $\text{GL}(\overline{H})$ -submodules of $\Phi(H)$ such that $\Phi(H) = H' \oplus H^p$. Hence K must stabilize $H^p N$

and so Lemma 5 gives that $K \leq \text{Aut}(G)^{\overline{G}}$. Since K is irreducible, so is $\text{Aut}(G)^{\overline{G}}$. As

$$\frac{\Phi(H)}{H^p N} = \frac{H^p H'}{H^p N} \cong \frac{H'}{N},$$

we obtain that K and $\text{Aut}(G)^{\overline{G}}$ are irreducible on $\Phi(H)/(H^p N)$. Now Theorem 6 implies that G is a UCS p -group with exponent p .

(b) The discussion at the beginning of this section shows that (b1) implies (b2) with $k = 1$. Before proving the converse, we argue that we may assume that $k = 1$. Suppose that V is an ESQ $\mathbb{F}_q K$ -module, and U is an $\mathbb{F}_q K$ -submodule satisfying $\Lambda^2 V/U \cong V$. Using the notation in part (a), the $\mathbb{F}_p(ZK)$ -homomorphism $\varepsilon: \Lambda_{\mathbb{F}_p}^2 V \rightarrow \Lambda_{\mathbb{F}_q}^2 V$ gives rise to an $\mathbb{F}_p(ZK)$ -isomorphism $\Lambda_{\mathbb{F}_p}^2 V/\hat{U} \cong \Lambda_{\mathbb{F}_q}^2 V/U$ where $\hat{U} := \varepsilon^{-1}(U)$. Since $\Lambda^2 V/U \cong V$ is an $\mathbb{F}_p K$ -isomorphism, it follows that $\Lambda_{\mathbb{F}_p}^2 V$ is ESQ. Moreover, V is an irreducible $\mathbb{F}_p K$ -module by [HB82, Theorem VII.1.16(e)]. In summary, viewing V as a K -module over \mathbb{F}_p of larger dimension allows us to assume that the hypotheses for (b2) hold for $k = 1$.

Suppose that $k = 1$. Set $H = H_{p,r}$. Take K to be an irreducible subgroup of $\text{GL}(\overline{H})$, and M to be a K -submodule of H' such that \overline{H} and H'/M are isomorphic. Specifically let $\varphi: \overline{H} \rightarrow H'/M$ be a K -module isomorphism. Set

$$N = \{x^p y \mid x \in H, y \in H' \text{ such that } \varphi(x\Phi(H)) = yM\},$$

and set $G = H/N$. As $p \geq 3$, one can easily check that N is a subgroup of $\Phi(H)$ and that $H' \cap N = M$. Lemma 4 shows that the map $x\Phi(H) \mapsto x^p$ is an isomorphism between the $\text{GL}(\overline{H})$ -modules \overline{H} and H^p . Therefore, if $g \in K$ and $x^p y \in N$ with some $x \in H$ and $y \in H'$ then, as φ is a K -homomorphism, $(x^p y)^g = (x^g)^p y^g \in N$. Thus N is a K -submodule. Therefore Lemma 5 shows that $K \leq \text{Aut}(G)^{\overline{G}}$. By assumption K is irreducible on $\overline{H} \cong \overline{G}$. The definition of N gives that $H'N = \Phi(H)$, and so

$$\frac{\Phi(H)}{N} = \frac{H'N}{N} \cong \frac{H'}{H' \cap N} = \frac{H'}{M} \cong \overline{G}.$$

As $\Phi(G) \cong \Phi(H)/N$, we obtain that K acts irreducibly on $\Phi(G)$. Since $K \leq \text{GL}(\overline{H})_N$, this shows that $(\text{GL}(\overline{H})_N)^{\Phi(H)/N}$ is irreducible. Hence, by Theorem 6, G must be a UCS p -group. Since $N \cap H^p = 1$, G has exponent p^2 . Thus (b2) implies (b1). \square

4. UCS p -GROUPS WITH GENERATOR NUMBER AT MOST 3

In this section we classify 2- and 3-generator UCS p -groups. The main result of this section is the following theorem from which Theorem 1(a)-(b) follows.

Theorem 8. *Let G be an r -generated non-abelian UCS p -group.*

- (a) *If $p = r = 2$, then G is isomorphic to the quaternion group Q_8 , and if $p = 2$ and $r = 3$, then $G \cong G_1$ where*

$$G_1 = \langle x_1, x_2, x_3 \mid x_1^2[x_1, x_2][x_1, x_3][x_2, x_3], x_2^2[x_1, x_2][x_1, x_3], x_3^2[x_1, x_2], 2\text{-class } 2 \rangle$$

has order 2^6 . Further, $\text{Aut}(Q_8)^{\overline{Q_8}} \cong \text{GL}_2(2)$ and $\text{Aut}(G_1)^{\overline{G_1}}$ has order 21.

- (b) *If $p \geq 3$ and $r = 2$, then $G \cong G_2$ where $G_2 = \langle x_1, x_2 \mid x_1^p, x_2^p, p\text{-class } 2 \rangle$ is extraspecial of order p^3 and exponent p . Further, $\text{Aut}(G_2)^{\overline{G_2}} \cong \text{GL}_2(p)$.*
- (c) *If $p \geq 3$ and $r = 3$, then G has order p^6 and $G \cong G_3$ or G_4 where*

$$G_3 = \langle x_1, x_2, x_3 \mid x_1^p, x_2^p, x_3^p, p\text{-class } 2 \rangle, \quad \text{and}$$

$$G_4 = \langle x_1, x_2, x_3 \mid x_1^p = [x_2, x_3], x_2^p = [x_3, x_1], x_3^p = [x_1, x_2], p\text{-class } 2 \rangle.$$

Further, $\text{Aut}(G_3)^{\overline{G_3}} \cong \text{GL}_3(p)$ and $\text{Aut}(G_4)^{\overline{G_4}} \cong \text{SO}_3(p)$.

Proof. (a) If $p = 2$ and $r = 2$, then $|G'| = 2$ and so $|G| = 8$. The dihedral group of order 8 has a characteristic cyclic subgroup of order 4, so the only possibility is that $G \cong Q_8$. The group Q_8 can be written as $H_{2,2}/N$ where $N = \langle x^2[y, x], y^2[y, x] \rangle$ and x, y are the generators of $H_{2,2}$. Now easy calculation shows that N is invariant under $\text{GL}(\overline{H_{2,2}})$, and so Lemma 5 implies that $\text{Aut}(Q_8)^{\overline{Q_8}} \cong \text{GL}_2(2)$. Let now $p = 2, r = 3$. It can be checked that every irreducible subgroup of $\text{GL}_3(2)$ has order divisible by 7. All subgroups of order 7 are conjugate in $\text{GL}_3(2)$, so we may take an arbitrary one. Its action on $\Phi(H_{2,3})$ is the sum of two non-isomorphic irreducible 3-dimensional submodules, say $(H_{2,3})'$ and N . Hence $H_{2,3}/N$ is a non-abelian UCS-group. A direct calculation (or an application of a computational algebra system [GAP07, BCP97]) shows that $H_{2,3}/N \cong G_1$, and $\text{Aut}(G_1)^{\overline{G_1}}$ is a non-abelian subgroup of $\text{GL}_3(2)$ of order 21.

(b) Suppose that p is odd, and $H = H_{p,2}$. Let N be a subgroup of $\Phi(H)$ such that $G = H/N$ is a non-abelian UCS p -group. As $G' = \Phi(G)$ has order p , it follows that $N = H^p$. Moreover, H/H^p is an exponent- p UCS extraspecial group isomorphic to G_2 . By Lemma 4, H^p is invariant under $\text{GL}(\overline{H})$, and so $\text{Aut}(G)^{\overline{G}} \cong \text{GL}_2(p)$.

This completes the proof of parts (a) and (b). The proof of part (c) relies on the following lemma.

Lemma 9. *Suppose that V is a 3-dimensional vector space over a finite field \mathbb{F} , where $\text{char}(\mathbb{F}) \neq 2$. Let U be a subspace of $V \oplus \Lambda^2 V$ such that $\dim U = 3, U \cap V = U \cap \Lambda^2 V = 0$*

and that $\mathrm{GL}(V)_U$ is irreducible on V . Then there exists a $g \in \mathrm{GL}(V)$ such that $Ug = W$ where

$$W = \langle e_1 - e_2 \wedge e_3, e_2 - e_3 \wedge e_1, e_3 - e_1 \wedge e_2 \rangle.$$

and e_1, e_2, e_3 is a basis for V . Further, $\mathrm{GL}(V)_U = g\mathrm{GL}(V)_W g^{-1}$ and $\mathrm{GL}(V)_W = \mathrm{SO}_3(\mathbb{F})$.

Proof. Let e_1, e_2, e_3 be a basis for V , and let $e_2 \wedge e_3, e_3 \wedge e_1, e_1 \wedge e_2$ be the corresponding dual basis for $\Lambda^2 V$. Concatenating these bases gives a basis for $V \oplus \Lambda^2 V$. We view $g \in \mathrm{GL}(V)$ as a 3×3 matrix relative to the basis e_1, e_2, e_3 . An easy computation shows that the transformation $g \wedge g \in \mathrm{GL}(\Lambda^2 V)$ defined by $(u \wedge v)(g \wedge g) = (ug) \wedge (vg)$, has matrix $\det(g)(g^{-1})^T = \det(g)g^{-T}$ relative to the above dual basis. Thus g acting on $V \oplus \Lambda^2 V$ has matrix

$$\begin{pmatrix} g & 0 \\ 0 & g \wedge g \end{pmatrix} = \begin{pmatrix} g & 0 \\ 0 & \det(g)g^{-T} \end{pmatrix}. \quad (1)$$

The subspace U has a basis of the form $e_1 - a_1, e_2 - a_2, e_3 - a_3$ where a_1, a_2, a_3 is a basis for $\Lambda^2 V$. We now calculate the stabilizer $\mathrm{GL}(V)_U$. Let U_A denote the 3×6 matrix $(I \mid -A)$ where A is the invertible 3×3 matrix with i th row

$$(a_{i1}, a_{i2}, a_{i3}) \quad \text{where} \quad a_i = a_{i1}e_2 \wedge e_3 + a_{i2}e_3 \wedge e_1 + a_{i3}e_1 \wedge e_2.$$

Note that the matrix $(I \mid -A)$ possesses two 3×3 sub-blocks. We shall view U as the row space of U_A . Let g be an arbitrary invertible 3×3 matrix. Then

$$(I \mid -A) \begin{pmatrix} g & 0 \\ 0 & g \wedge g \end{pmatrix} = (g \mid -A(g \wedge g))$$

However, the matrices $(g \mid -A(g \wedge g))$ and $(I \mid -g^{-1}A(g \wedge g))$ have the same row space. The image of U_A under the $\mathrm{GL}(V)$ -action is thus $(U_A)g = U_{g^{-1}A(g \wedge g)} = U_{\det(g)g^{-1}Ag^{-T}}$, by equation (1). Hence $g \in \mathrm{GL}(V)$ stabilizes $U = U_A$ if and only if $A = g^{-1}A(g \wedge g)$, or A intertwines g and $g \wedge g$. In summary, $g \in \mathrm{GL}(V)_U$ if and only if $gAg^T = (\det g)A$.

The stabilizer $\mathrm{GL}(V)_U$ is contained in the subgroup

$$\Gamma = \{g \in \mathrm{GL}(V) \mid g(A - A^T)g^T = (\det g)(A - A^T)\}.$$

However Γ , and hence $\mathrm{GL}(V)_U$, fixes the null space

$$\{v \in V \mid v(A - A^T) = 0\}.$$

As $\mathrm{GL}(V)_U$ acts irreducibly on V , $A - A^T$ is either 0 or invertible. Since

$$\det(A - A^T) = \det((A - A^T)^T) = \det(A^T - A) = (-1)^3 \det(A - A^T)$$

and $\text{char}(\mathbb{F}) \neq 2$, we see that $\det(A - A^T) = 0$. Thus $A - A^T = 0$, and A is symmetric. Given that A is invertible, $gAg^T = (\det g)A$ for $g \in \text{GL}(V)_U$ implies $\det(g) = 1$. In summary, $\text{GL}(V)_U$ is the special orthogonal group

$$\text{GL}(V)_U = \{g \in \text{GL}(V) \mid gAg^T = A \text{ and } \det(g) = 1\}.$$

In particular, if $A = I$ and $W = U_I$, then

$$\text{GL}(V)_W = \{g \in \text{GL}(V) \mid gg^T = I \text{ and } \det(g) = 1\} = \text{SO}_3(\mathbb{F}).$$

The quadratic form $Q_A: V \rightarrow \mathbb{F}: v \mapsto \frac{1}{2}vAv^T$ determines (and is determined by) a non-degenerate symmetric bilinear form $\beta_A: V \times V \rightarrow \mathbb{F}: (v, w) \mapsto vAw^T$. By diagonalizing β_A (see [Lam73, Ch. 1, Cor. 1.2.4]), there exists $g_1 \in \text{GL}(V)$ such that $g_1^{-1}A(g_1^{-1})^T$ is a non-zero scalar matrix. Thus $(U_A)g_1 = U_{g_1^{-1}Ag_1^{-T}\det(g_1)} = U_{\lambda I}$ where λI is a non-zero scalar matrix. However, $(U_{\lambda I})(\lambda^{-1}I) = U_I$. Thus $U_Ag = U_I = W$ where $g = g_1\lambda^{-1}$. \square

The proof of Theorem 8(c). Let p be an odd prime, set $H = H_{p,3}$ and let $G = H/N$ be a 3-generator UCS p -group. By Lemma 4, the $\text{GL}(\overline{H})$ -modules \overline{H} and H^p are equivalent via the p -th power map. The action of $\text{GL}(\overline{H})$ on H' is equivalent to the exterior square action. In the proof of Lemma 9, the action of $\text{GL}(V)$ on $\Lambda^2 V$ was shown to be $g \mapsto \det(g)(g^{-1})^T$. Thus by Lemma 4 and Theorem 6 the stabilizer $K := \text{GL}(\overline{H})_N$ acts irreducibly on $\overline{H} \cong H^p \cong V$ and on $H' \cong \Lambda^2 V$.

If G has exponent p , then $H^p \leq N$. By Lemma 4, H' is invariant under $\text{GL}(\overline{H})$, and so the subspace $N \cap H'$ must be invariant under $\text{GL}(\overline{H})_N$. If $1 < N \cap H' < H'$, then $\text{GL}(\overline{H})_N$ is reducible on H' contradicting the previous paragraph. Thus, by Theorem 6, $N \cap H' = 1$, and $G = H/H^p$. Hence G must be isomorphic to the group G_3 in the statement of the theorem, and $\text{Aut}(G_3)^{\overline{G_3}} \cong \text{GL}_3(p)$.

Suppose now that G has exponent p^2 . By Theorem 6(iii), $|N| = p^3$, $N \cap H^p = 1$ and $\text{GL}(\overline{H})_N$ is irreducible. As the $\text{GL}(\overline{H})$ -actions on H' and $\Lambda^2 \overline{H}$ are equivalent, Lemma 9 with $\mathbb{F} = \mathbb{F}_p$ shows that a minimal generating set x_1, x_2, x_3 of G can be chosen satisfying the relations of G_4 . Lemma 9 also yields that $\text{Aut}(G_4)^{\overline{G_4}} \cong \text{SO}_3(p)$. As $\text{SO}_3(p)$ acts irreducibly on both G/G^p and G^p , we see that $G = G_4$ is a UCS-group, as required. \square

5. 4-GENERATOR UCS 2-GROUPS

In this brief section we describe a computer-based classification of 4-generator UCS 2-groups. Recall that $\{x_1, x_2, x_3, x_4\}$ is a fixed minimal generating set for $H_{2,4}$. Let

y_1, y_2, y_3, y_4 denote the squares $x_1^2, x_2^2, x_3^2, x_4^2$, and let $z_1, z_2, z_3, z_4, z_5, z_6$ denote the commutators $[x_1, x_2], [x_1, x_3], [x_1, x_4], [x_2, x_3], [x_2, x_4], [x_3, x_4]$ in $H_{2,4}$, respectively. Each group below has the form $H_{2,4}/N$ where N is a subgroup of the Frattini subgroup

$$\Phi(H_{2,4}) = \langle y_1, y_2, y_3, y_4, z_1, z_2, z_3, z_4, z_5, z_6 \rangle.$$

The following theorem proves the part of Theorem 1(c) with $p = 2$. The homocyclic abelian group $H_{2,4}/N_5$ below is not included in Theorem 1(c).

Theorem 10. *A 4-generator UCS 2-group is isomorphic to the group $H_{2,4}/N$ where N is precisely one of the 9 subgroups described below:*

$$\begin{aligned} N_1 &= \langle y_1, y_2, y_3, y_4, z_1 z_3, z_2, z_3 z_4, z_5, z_6 \rangle; & N_2 &= \langle y_1, y_2 y_3, y_3 z_4, y_4, z_1 z_3, z_2, z_3 z_4, z_5, z_6 \rangle; \\ N_3 &= \langle y_1 z_1, y_2 z_1, y_3, y_4, z_1 z_2 z_3, z_2 z_3 z_5, z_3 z_4, z_6 \rangle; \\ N_4 &= \langle y_1 z_1, y_2 z_2, y_3 z_2, y_4 z_1, z_1 z_5, z_2 z_3 z_5, z_3 z_4, z_6 \rangle; \\ N_5 &= \langle z_1, z_2, z_3, z_4, z_5, z_6 \rangle; & N_6 &= \langle y_1 z_3, y_2, y_3, y_4, z_1 z_5, z_6 \rangle; \\ N_7 &= \langle y_1 z_2, y_2 z_5, y_3, y_4, z_1 z_6, z_2 z_5 z_6 \rangle; & N_8 &= \langle y_1 z_2 z_4, y_2 y_4 z_3, y_3 y_4 z_4, y_4 z_1, z_1 z_6, z_2 z_5 z_6 \rangle; \\ N_9 &= \langle y_1 z_3, y_2 z_4, y_3 z_4, y_4 z_3, z_1 z_6, z_2 z_5 z_6 \rangle. \end{aligned}$$

Proof. The proof relies primarily on computer calculations. One can easily verify, using the command `CharacteristicSubgroups` in `GAP` [GAP07], that each of the above 9 quotient groups are UCS 2-groups. Clearly, they all are 4-generator groups.

Suppose that G is a 4-generator UCS 2-group. Then $G \cong H_{2,4}/N$ where $N \leq \Phi(H_{2,4})$ by Lemma 3. If G is abelian, then G^2 is the non-trivial and proper characteristic subgroup of G . In this case, G must be isomorphic to the homocyclic group $H_{2,4}/N_5 \cong (C_4)^4$.

Assume now that G is non-abelian. As $\Phi(G) = G'$ and $|(H_{2,4})'| = 2^6$, we deduce that $|G| \leq 2^{10}$. The classification of 2-groups with order at most 2^{10} is part of the computational algebra systems `MAGMA` and `GAP` [BCP97, GAP07]. Both systems can be used to determine whether a given 2-group is a 4-generator UCS group. Although the system `MAGMA` was faster than `GAP`, it was still unable to deal with the large number of groups of order 2^{10} . Suppose that H is a 4-generator group in the `MAGMA` library with order dividing 2^9 where $H' = \Phi(H) = Z(H)$. We used `MAGMA` to compute $\text{Aut}(H)$, and checked whether $\text{Aut}(H)^{\overline{H}}$ and $\text{Aut}(H)^{\Phi(H)}$ are irreducible linear groups. By Theorem 6, H is UCS if and only if both of these linear groups are irreducible. In this way one can verify that if $|G|$ divides 2^9 , then G is precisely one of the 9 quotient groups above. Looping over the groups with order 2^9 on a computer with a 1.8 GHz CPU and 512 MB

memory took approximately 20 CPU minutes. A similar computation for $|G| = 2^{10}$ never completed.

It remains to show that no 4-generator UCS group $G = H_{2,4}/N$ has order 2^{10} where $N \leq \Phi(H_{2,4})$ has order $|N| = 2^4$. By Theorem 6, $S := \text{GL}_4(2)_N$ is an irreducible subgroup of $\text{GL}_4(2)$. View N as a subspace of $\Phi(H_{2,4}) \cong (\mathbb{F}_2)^{10}$, and identify $\Phi(H_{2,4})/(H_{2,4})'$ with $V = (\mathbb{F}_2)^4$, and $(H_{2,4})'$ with $\Lambda^2 V$. As $|G'| = |\Lambda^2 V| = 2^6$, we see that $N \cap \Lambda^2 V = 0$, and so $\Phi(H_{2,4})$ admits the S -module direct decomposition $\Phi(H_{2,4}) = N \oplus \Lambda^2 V$. Further, since $\Phi(H_{2,4})/N \cong \Lambda^2 V$, the action of S on $\Lambda^2 V$ is irreducible by Theorem 6. We use MAGMA to loop over all irreducible subgroups R of $\text{GL}(V) \cong \text{GL}_4(2)$. We find that either R is reducible on $\Lambda^2 V$, or the R -action on $\Phi(H_{2,4})$ fixes no 4-dimensional subspace that could correspond to N . This contradiction proves that no such group G exists. \square

The GAP and MAGMA catalogue numbers of the 9 quotient groups in Theorem 10 are [32, 49], [32, 50], [64, 242], [64, 245], [256, 6732], [256, 8935], [256, 10090], [256, 10289], and [256, 10297], respectively. The 1-dimensional semilinear group $\Gamma\text{L}_1(16)$ has a presentation $\langle a, b \mid a^4 = b^{15} = 1, b^a = b^2 \rangle$. The automorphism groups of the 9 USC 2-groups induce on the Frattini quotient the following irreducible subgroups of $\text{GL}_4(2)$: $\text{O}_4^+(2)$, $\text{O}_4^-(2)$, $\text{GL}_2(2) \boxtimes \text{GL}_2(2) \cong S_3 \times S_3$, $\Gamma\text{L}_1(16)$, $\text{GL}_4(2)$, $\text{O}_4^+(2)$, $\langle a, b^3 \rangle$, $\langle b^3 \rangle$, and $\Gamma\text{L}_2(4)$ respectively. This can be deduced by using GAP or MAGMA to compare the chief series of irreducible subgroups of $\text{GL}_4(2)$, and the chief series of automorphism groups of USC 2-groups.

The groups in Theorem 10 indicate that an important class of UCS 2-groups are formed by the Suzuki 2-groups. Computation with GAP [GAP07] shows that the group $H_{2,4}/N_4$ is isomorphic to the Suzuki 2-group \mathcal{Q} described as a (3×3) -matrix group in [HB82, VIII.7.10 Remark] with $n = 4$. Further, the group $H_{2,4}/N_9$ is isomorphic to the group $A(4, \vartheta)$ in [HB82, 6.7 Example] where ϑ is the Frobenius automorphism of \mathbb{F}_{16} . However, since ϑ has order 4, by [HB82, 6.9 Theorem], this group is not a Suzuki 2-group. In a Suzuki 2-group G , the subgroup $\Phi(G)$ coincides with the elements of order at most 2 [HB82, VII.7.9 Theorem], and so the automorphism ξ that permutes the involutions transitively acts irreducibly on $\Phi(G)$. Moreover, it is noted in the proof of [HB82, 7.9 Theorem(b)] that in a Suzuki 2-group G of type $A(\vartheta, n)$ the automorphism ξ that permutes the set of involutions transitively acts irreducibly on \overline{G} , and so Theorem 6 implies that these groups are always UCS groups. We claim, in addition, that the group \mathcal{Q} in [HB82, VIII.7.10 Remark] is always UCS. Indeed, the group of diagonal matrices with z^{1-q}, z, z^2 in the diagonal where z runs through the elements of \mathbb{F}_{q^2} induces a Singer

cycle on the quotient $\mathcal{Q}/\overline{\mathcal{Q}}$, and hence this group is irreducible on $\mathcal{Q}/\overline{\mathcal{Q}}$. Thus Theorem 6 gives that \mathcal{Q} is UCS. Though we conjecture that Suzuki 2-groups are always UCS groups, the detailed investigation of these groups goes beyond the scope of this paper.

6. 4-GENERATOR UCS p -GROUPS WITH EXPONENT p

Suppose that p is odd. In this section we classify 4-generator UCS p -groups with exponent p . That is, we prove the following theorem.

Theorem 11. *Let p be an odd prime, let $\{x_1, x_2, x_3, x_4\}$ be a minimal generating set for $H_{p,4}$. Fix $\alpha \in \mathbb{F}_p^\times$ such that $\mathbb{F}_p^\times = \langle -\alpha \rangle$. The following is a complete and irredundant list of the isomorphism classes of 4-generator UCS p -groups with exponent p :*

- (i) $G_0 = H_{p,4}/(H_{p,4})^p$;
- (ii) $G_2 = H_{p,4}/\langle (H_{p,4})^p, [x_1, x_2][x_3, x_4]^{-1}, [x_1, x_3], [x_1, x_4], [x_2, x_3], [x_2, x_4] \rangle$;
- (iii) $G_4 = H_{p,4}/\langle (H_{p,4})^p, [x_1, x_3], [x_1, x_4], [x_2, x_3], [x_2, x_4] \rangle$;
- (iv) $G_6 = H_{p,4}/\langle (H_{p,4})^p, [x_1, x_2], [x_3, x_4], [x_2, x_3][x_1, x_4]^{-1}, [x_1, x_3]^\alpha[x_2, x_4] \rangle$;
- (v) $G_{11} = H_{p,4}/\langle (H_{p,4})^p, [x_1, x_4], [x_2, x_3], [x_2, x_4][x_1, x_3]^{-1} \rangle$;
- (vi) $G_{14} = H_{p,4}/\langle (H_{p,4})^p, [x_1, x_2][x_3, x_4] \rangle$;
- (vii) $G_{16} = H_{p,4}/\langle (H_{p,4})^p, [x_1, x_2], [x_3, x_4] \rangle$;
- (viii) $G_{18} = H_{p,4}/\langle (H_{p,4})^p, [x_2, x_3][x_1, x_4], [x_1, x_3]^\alpha[x_2, x_4]^{-1} \rangle$.

The indexing of groups in Theorem 11 is explained later. It is related to the 18 orbits of $\text{GL}(V)$ on the proper non-trivial subspaces of $\Lambda^2 V$.

We recall that $H_{p,r}$ is defined in Section 2. Theorem 11 is the quantitative version of Theorem 1(c) and its proof relies on the classification of 4-generator p -groups with exponent p and nilpotency class 2. In order to present the classification here, we need some notation. Since p is odd, a 4-generator p -group G with nilpotency class 2 and exponent p is isomorphic to $H_{p,4}/((H_{p,4})^p N)$ where N is a subgroup of $(H_{p,4})'$. Since $(H_{p,4})'$ is an elementary abelian group, we view it as a vector space over \mathbb{F}_p . Assume that $H_{p,4}$ is generated by x_1, x_2, x_3, x_4 . Consider the subgroup $(H_{p,4})'$ as a 6-dimensional subspace over \mathbb{F}_p with standard basis $[x_1, x_2], [x_1, x_3], [x_1, x_4], [x_2, x_3], [x_2, x_4], [x_3, x_4]$. We introduce a non-degenerate quadratic form on $(H_{p,4})'$:

$$Q'([x_1, x_2]^{\alpha_1}[x_1, x_3]^{\alpha_2}[x_1, x_4]^{\alpha_3}[x_2, x_3]^{\alpha_4}[x_2, x_4]^{\alpha_5}[x_3, x_4]^{\alpha_6}) = \alpha_1\alpha_6 - \alpha_2\alpha_5 + \alpha_3\alpha_4.$$

The quadratic form Q' induces a non-degenerate symmetric bilinear form:

$$(v_1, v_2) = Q'(v_1 + v_2) - Q'(v_1) - Q'(v_2).$$

If U is a subgroup in $(H_{p,4})'$, then U^\perp is defined as

$$U^\perp = \{v \in (H_{p,4})' \mid (u, v) = 0 \text{ for all } u \in U\}.$$

Let α be as in Theorem 11, and define the following subgroups in $(H_{p,4})'$:

- $N_0 = 1$;
- $N_1 = \langle [x_1, x_3], [x_1, x_4], [x_2, x_3], [x_2, x_4], [x_3, x_4] \rangle$;
- $N_2 = \langle [x_1, x_2][x_3, x_4]^{-1}, [x_1, x_3], [x_1, x_4], [x_2, x_3], [x_2, x_4] \rangle$;
- $N_3 = \langle [x_1, x_2], [x_1, x_4], [x_2, x_4], [x_3, x_4] \rangle$;
- $N_4 = \langle [x_1, x_3], [x_1, x_4], [x_2, x_3], [x_2, x_4] \rangle$;
- $N_5 = \langle [x_1, x_2], [x_2, x_4], [x_3, x_4], [x_2, x_3][x_1, x_4]^{-1} \rangle$;
- $N_6 = \langle [x_1, x_2], [x_3, x_4], [x_2, x_3][x_1, x_4]^{-1}, [x_1, x_3]^\alpha[x_2, x_4] \rangle$;
- $N_7 = \langle [x_1, x_4], [x_2, x_4], [x_3, x_4] \rangle$;
- $N_8 = \langle [x_1, x_3], [x_1, x_4], [x_2, x_4] \rangle$;
- $N_9 = \langle [x_2, x_3], [x_2, x_4], [x_3, x_4] \rangle$;
- $N_{10} = \langle [x_1, x_3], [x_1, x_4], [x_3, x_4][x_1, x_2]^{-1} \rangle$;
- $N_{11} = \langle [x_1, x_4], [x_2, x_3], [x_2, x_4][x_1, x_3]^{-1} \rangle$;
- $N_{12} = \langle [x_1, x_4], [x_2, x_4][x_1, x_3]^{-\alpha}, [x_1, x_2][x_3, x_4]^{-1} \rangle$.

For $i = 13, \dots, 18$ set $N_i = (N_{i-12})^\perp$. That is, N_i is the subgroup perpendicular to N_{i-12} with respect to the symmetric bilinear form $(\ , \)$ associated to Q' . For $i = 0, \dots, 18$, let G_i denote the group $H_{p,4}/((H_{p,4})^p N_i)$. Our notation is consistent in the sense that the groups G_i defined here coincide with those defined in Theorem 11.

Lemma 12. *For $p \geq 3$, every 4-generator finite p -group with exponent p and nilpotency class 2 is isomorphic to precisely one of the groups G_0, G_1, \dots, G_{18} .*

Proof. Suppose that $i \in \{0, \dots, 5\}$ and let \mathcal{N}_i denote the set of subgroups with order p^i in $(H_{p,4})'$. Then, for $M_1, M_2 \in \mathcal{N}_i$, we have $H_{p,4}/((H_{p,4})^p M_1) \cong H_{p,4}/((H_{p,4})^p M_2)$ if and only if M_1 and M_2 lie in the same $\text{GL}(\overline{H_{p,4}})$ -orbit (see [O'B90, Theorem 2.8]). Therefore it suffices to show that the subgroups N_0, \dots, N_{18} form a complete and irredundant set of representatives of the $\text{GL}(\overline{H_{p,4}})$ -orbits in $\bigcup \mathcal{N}_i$. A simple computation shows that Q' is stabilized by $\text{GL}(\overline{H_{p,4}})$ up to scalar multiples (alternatively see [KL90, Proposition 2.9.1(vii)]). Thus M_1 and M_2 lie in the same $\text{GL}(\overline{H_{p,4}})$ -orbit if and only if

$(M_1)^\perp$ and $(M_2)^\perp$ do. Therefore we only need to verify that the set $\{N_1, \dots, N_{12}\}$ is a set of representatives of the $\text{GL}(\overline{H_{p,4}})$ -orbits in $\mathcal{N}_5 \cup \mathcal{N}_4 \cup \mathcal{N}_3$. These orbits have long been known; they are listed already in Brahana's paper [Bra40]. Our list above is taken from the recent classification of finite p -groups of order dividing p^7 , see [NOVL04, OVL05]. \square

Recall that $\{x_1, x_2, x_3, x_4\}$ is a fixed generating set for $H_{p,4}$. Risking confusion, let V denote a 4-dimensional vector space over \mathbb{F}_p with basis x_1, x_2, x_3, x_4 . This way the symbol x_i may refer to an element of $H_{p,4}$, or to an element of V . However, elements of the group $H_{p,4}$ are written multiplicatively, while elements of V are written additively. There is a natural bijection $\Psi : \Lambda^2 V \rightarrow (H_{p,4})'$ mapping $x_i \wedge x_j \mapsto [x_i, x_j]$. Define a quadratic form Q on $\Lambda^2 V$ by $Q(w) = Q'(\Psi(w))$ for all $w \in \Lambda^2 V$. The value of the form Q is given by:

$$\begin{aligned} Q(\alpha_1 x_1 \wedge x_2 + \alpha_2 x_1 \wedge x_3 + \alpha_3 x_1 \wedge x_4 + \alpha_4 x_2 \wedge x_3 + \alpha_5 x_2 \wedge x_4 + \alpha_6 x_3 \wedge x_4) \\ = \alpha_1 \alpha_6 - \alpha_2 \alpha_5 + \alpha_3 \alpha_4. \end{aligned}$$

Theorem 6 says that a subspace $U \leq \Lambda^2 V$ gives rise to an exponent- p UCS p -group $G_U := H_{p,4}/((H_{p,4})^p \Psi(U))$ if and only if the stabilizer $\text{GL}(V)_U$ is irreducible on both V and $\Lambda^2 V/U$. For $i = 0, \dots, 18$, let U_i denote the subspace $\Psi^{-1}(N_i)$. Therefore we can check which of the groups G_0, \dots, G_{18} are UCS by checking, for $i = 0, \dots, 18$, whether $\text{GL}(V)_{U_i}$ is irreducible on V and on $\Lambda^2 V/U_i$. This is carried out in the rest of this section.

A subspace $U \leq \Lambda^2 V$ is said to be *degenerate* if $U \cap U^\perp \neq 0$; otherwise it is said to be *non-degenerate*. A subspace U is said to be *totally isotropic* if $Q(u) = 0$ for all $u \in U$.

Lemma 13. *If U is a degenerate subspace of $\Lambda^2 V$, then $\text{GL}(V)_U$ acts reducibly on V or $\Lambda^2 V/U$, and thus the p -group $G_U := H_{p,4}/((H_{p,4})^p \Psi(U))$ is not a UCS-group. Further, the subspaces U_i for $i \in \{1, 3, 5, 7, 8, 9, 10, 12, 13, 15, 17\}$ are degenerate with respect to Q .*

Proof. Suppose that U is degenerate. Then $U \cap U^\perp \neq 0$, and $U + U^\perp$ is a proper subspace of $\Lambda^2 V$. Thus if $U^\perp \not\leq U$, then $\text{GL}(V)_U = \text{GL}(V)_{U^\perp}$ is reducible on $\Lambda^2 V/U$ as it fixes the proper subspace $(U + U^\perp)/U$. Assume henceforth that $U^\perp \leq U$.

Lemma 12 says that there are precisely 19 different $\text{GL}(V)$ -orbits on the proper subspaces of $\Lambda^2 V$, and representatives of the orbits are U_0, U_1, \dots, U_{18} . A straightforward calculation shows that U_i is degenerate if and only if $i \in \{1, 3, 5, 7, 8, 9, 10, 12, 13, 15, 17\}$. Moreover, the only U_i satisfying $U^\perp \leq U$ are U_1, U_3, U_7, U_9 . It is possible to prove case by case that for these values of i , the group G_{U_i} is not UCS. However, instead of performing a case by case analysis, we offer a geometric proof, which we find more elegant.

We shall show that for each degenerate subspace $U \leq \Lambda^2 V$ with $U^\perp \leq U$, the stabilizer $\mathrm{GL}(V)_{U^\perp} = \mathrm{GL}(V)_U$ is a reducible subgroup of $\mathrm{GL}(V)$. Since $\dim(U^\perp) = 6 - \dim(U)$, we see that $\dim(U^\perp) = 1, 2, 3$ and U^\perp is a totally isotropic subspace of $\Lambda^2 V$. First, suppose that $\dim(U^\perp) = 1$. By [Tay92, p.187], there exist linearly independent vectors $v_1, v_2 \in V$ such that $U^\perp = \langle v_1 \wedge v_2 \rangle$ and hence $\mathrm{GL}(V)_{U^\perp}$ fixes $\langle v_1, v_2 \rangle$ and so is reducible. Second, suppose that $\dim(U^\perp) = 2$. Then also by [Tay92, Lemma 12.15], U^\perp has the form $\langle v_1 \wedge v_3, v_2 \wedge v_3 \rangle$ where $v_1, v_2, v_3 \in V$ are linearly independent. Thus $\mathrm{GL}(V)_{U^\perp}$ fixes $\langle v_1, v_2, v_3 \rangle$. Finally, suppose that $\dim(U^\perp) = 3$, then by [Tay92, Theorem 12.16], U^\perp equals $W_1 \wedge V$ or $W_3 \wedge W_3$ where W_1, W_3 are subspaces of V of dimension 1 and 3 respectively. Thus $\mathrm{GL}(V)_{U^\perp}$ fixes W_1 or W_3 , and so is reducible. \square

The proof of the converse of Lemma 13 is substantially harder. It is noteworthy that in the proof of Lemma 14 below, $\mathrm{Aut}(G)^{\overline{G}}$ is commonly a maximal group from one of Aschbacher's [Asc84] classes.

Lemma 14. *If U is a non-degenerate subspace of $\Lambda^2 V$, then $\mathrm{GL}(V)_U$ acts irreducibly on V and $\Lambda^2 V/U$, and thus the p -group $G_U := H_{p,4}/((H_{p,4})^p \Psi(U))$ is a UCS-group. Further, the subspaces U_i for $i \in \{0, 2, 4, 6, 11, 14, 16, 18\}$ are non-degenerate with respect to Q .*

Proof. As remarked in the proof of Lemma 13, U_0, U_1, \dots, U_{18} is a complete and irredundant list of representatives of $\mathrm{GL}(V)$ -orbits on the proper subspaces of $\Lambda^2 V$. Moreover, U_i is non-degenerate if and only if $i \in \{0, 2, 4, 6, 11, 14, 16, 18\}$. Denote the stabilizer $\mathrm{GL}(V)_{U_i}$ by K_i . We prove below, in a case by case manner, that K_i acts irreducibly on V and $\Lambda^2 V/U_i$ for $i \in \{0, 2, 4, 6, 11, 14, 16, 18\}$.

Since $\mathrm{GL}(V)$ is irreducible on V and $\Lambda^2 V$, G_0 is a UCS-group. Suppose henceforth that $U_i \neq 0$. For non-degenerate U , $\Lambda^2 V = U \oplus U^\perp$ and $\mathrm{GL}(V)_U = \mathrm{GL}(V)_{U^\perp}$. Since $U_{i+12} = U_i^\perp$ for $1 \leq i \leq 6$, we see that $K_i = K_{i+12}$. Thus it suffices to prove that K_i is irreducible on the spaces V , U_i , and $U_i^\perp \cong \Lambda^2 V/U_i$ for $i \in \{2, 4, 6, 11\}$.

Consider first the stabilizer K_2 . Note that $U_2^\perp = U_{14} = \langle x_1 \wedge x_2 + x_3 \wedge x_4 \rangle$. View $g \in \mathrm{GL}(V)$ as a 4×4 matrix with respect to the basis x_1, x_2, x_3, x_4 of V . The equation

$$(x_1 \wedge x_2 + x_3 \wedge x_4)(g \wedge g) = \alpha(x_1 \wedge x_2 + x_3 \wedge x_4) \quad \text{for some non-zero } \alpha \in \mathbb{F}_p,$$

gives a linear system equivalent to the matrix equation $g^T J g = \alpha J$ where

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Thus $g \in K_2$ if and only if g preserves the alternating form J up to a scalar factor. In other words, $K_2 \cong \mathrm{GSp}_4(p)$. Clearly, $\mathrm{GSp}_4(p)$ is irreducible on V , and on the 1-dimensional subspace U_{14} . Additionally, the K_2 -action on U_2 is irreducible as K_2 contains a subgroup isomorphic to $\Omega_5(p)$ (see [KL90, Proposition 2.9.1(vi)]), and $\Omega_5(p)$ acts irreducibly on the 5-dimensional space U_2 .

Consider now the stabilizer K_4 . Set $L_1 = \langle x_1, x_2 \rangle$ and $L_2 = \langle x_3, x_4 \rangle$. Let H be the stabilizer of the decomposition $V = L_1 \oplus L_2$. Then $H \cong \mathrm{GL}_2(p) \wr C_2$. We aim to prove that $H = K_4$. It is routine to check that H stabilizes $U_{16} = \langle x_1 \wedge x_2, x_3 \wedge x_4 \rangle$ and $U_{16}^\perp = U_4$. Thus $H \leq K_4$. Set $P_1 = \langle x_1 \wedge x_2 \rangle$ and $P_2 = \langle x_3 \wedge x_4 \rangle$. Then $U_{16} = P_1 \oplus P_2$ and P_1, P_2 are the only 1-dimensional totally isotropic subspaces of U_{16} . Thus K_4 permutes the set $\{P_1, P_2\}$. The following argument shows that $\mathrm{GL}(V)_{P_1} = \mathrm{GL}(V)_{L_1}$:

$$\begin{aligned} g \in \mathrm{GL}(V)_{P_1} &\iff (x_1 \wedge x_2)(g \wedge g) = \alpha(x_1 \wedge x_2) \quad \text{for some } \alpha \in \mathbb{F}_p^\times \\ &\iff \langle x_1, x_2 \rangle g = \langle x_1, x_2 \rangle \\ &\iff g \in \mathrm{GL}(V)_{L_1}. \end{aligned}$$

Denote by \tilde{H} the subgroup of H that fixes both L_1 and L_2 , and denote by \tilde{K}_4 the subgroup of K_4 that fixes both P_1 and P_2 . Since $\mathrm{GL}(V)_{L_1} = \mathrm{GL}(V)_{P_1}$, we see that $\tilde{H} = \tilde{K}_4$. Since $|K_4 : \tilde{K}_4| = 2$ and $|H : \tilde{H}| = 2$, it follows that $|H| = |K_4|$. Thus $H = K_4$ as desired.

We claim that K_4 is irreducible on V , U_4 , and on U_{16} . The only proper and non-trivial \overline{K}_4 -submodules of V are L_1 and L_2 . These are swapped by K_4 , and so K_4 is irreducible on V . Suppose that $g \in \overline{K}_4$ is represented by the block-matrix

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

Then the action of g on U_{16} relative to the basis $x_1 \wedge x_2, x_3 \wedge x_4$ has matrix

$$\begin{pmatrix} \det A & 0 \\ 0 & \det B \end{pmatrix}.$$

Therefore the only proper, non-trivial \overline{K}_4 -submodules of U_{16} are P_1 and P_2 . However, K_4 contains an element that swaps P_1 and P_2 , and thus K_4 is irreducible on U_{16} . Recall that $U_4 = \langle x_1 \wedge x_3, x_1 \wedge x_4, x_2 \wedge x_3, x_2 \wedge x_4 \rangle$. The action of \overline{K}_4 on U_4 is equivalent to the action of $\mathrm{GL}_2(p) \times \mathrm{GL}_2(p)$ on the tensor product $\langle x_1, x_2 \rangle \otimes \langle x_3, x_4 \rangle$; the equivalence is realized by the map $x_i \wedge x_j \mapsto x_i \otimes x_j$. Since $\mathrm{GL}_2(p)$ is absolutely irreducible, [Rob96, 8.4.2] can be used to obtain that the outer tensor product $\mathrm{GL}_2(p) \boxtimes \mathrm{GL}_2(p)$ acts irreducibly on \mathbb{F}_p^4 . Hence \overline{K}_4 , and therefore K_4 , acts irreducibly on U_4 .

The next stabilizer to consider is K_6 . We shall show that K_6 acts irreducibly on V , U_6 and $U_{18} = U_6^\perp$. The definition of the subspace U_6 involves a fixed $\alpha \in \mathbb{F}_p^\times$ such that $-\alpha$ generates \mathbb{F}_p^\times . Note that

$$U_{18} = U_6^\perp = \langle x_1 \wedge x_4 + x_2 \wedge x_3, \alpha x_1 \wedge x_3 - x_2 \wedge x_4 \rangle. \quad (2)$$

We shall prove that $K_6 = K_{18} \cong \Gamma\mathrm{L}_2(p^2)$. Identify V with $\mathbb{F}_{p^2} \oplus \mathbb{F}_{p^2}$ as follows. Since $-\alpha$ is a non-square in \mathbb{F}_p , its square roots, $\pm\beta$, lie in $\mathbb{F}_{p^2} \setminus \mathbb{F}_p$. Then $(1, 0), (\beta, 0), (0, 1), (0, \beta)$ is an \mathbb{F}_p -basis of $\mathbb{F}_{p^2} \oplus \mathbb{F}_{p^2}$. Identify the elements x_1, x_2, x_3, x_4 with $(1, 0), (\beta, 0), (0, 1), (0, \beta)$, respectively. We view $\Gamma\mathrm{L}_2(p^2)$ as a subgroup of $\mathrm{GL}_4(p)$ under the identification above.

Let $\varepsilon : \Lambda_{\mathbb{F}_p}^2 V \rightarrow \Lambda_{\mathbb{F}_{p^2}}^2 V$ be the unique \mathbb{F}_p -linear map satisfying $\varepsilon(u \wedge v) = u \wedge v$. Easy computation shows that $U_6 \leq \ker \varepsilon$. On the other hand, by dimension counting, we have that $\dim(\ker \varepsilon) = 4$, which gives $U_6 = \ker \varepsilon$. Suppose that U is a 2-dimensional \mathbb{F}_p -subspace in V . Then U is an \mathbb{F}_{p^2} -subspace of V if and only if $\varepsilon(U \wedge U) = 0$; that is, $U \wedge U \leq U_6$. Thus K_6 is the setwise stabilizer of the \mathbb{F}_{p^2} -subspaces of V .

Clearly, the group $\Gamma\mathrm{L}_2(p^2)$ permutes the \mathbb{F}_{p^2} subspaces of V , and hence $\Gamma\mathrm{L}_2(p^2) \leq K_6$. In order to show the other direction, let $g \in \mathrm{GL}_4(p)$ and suppose that g stabilizes U_6 . Then g permutes the \mathbb{F}_{p^2} -subspaces. Thus we have, for $v \in V$ and $\alpha \in \mathbb{F}_{p^2}$, that

$$(\alpha v)g = \beta(vg) \quad (3)$$

with some $\beta \in \mathbb{F}_{p^2}$. Let $v_1, v_2 \in V$ be linearly independent over \mathbb{F}_{p^2} . Then, as g permutes the \mathbb{F}_{p^2} -subspaces, v_1g and v_2g are linearly independent over \mathbb{F}_{p^2} . Let $\beta_1, \beta_2 \in \mathbb{F}_{p^2}$ such that $(\alpha v_1)g = \beta_1(v_1g)$ and $(\alpha v_2)g = \beta_2(v_2g)$. Then there is some $\gamma \in \mathbb{F}_{p^2}$ such that $(\alpha v_1 + \alpha v_2)g = \gamma(v_1g + v_2g)$, but also $(\alpha v_1 + \alpha v_2)g = \beta_1(v_1g) + \beta_2(v_2g)$. This shows that $\beta_1 = \beta_2$, and therefore β is independent of v in equation (3). Hence for all $\alpha \in \mathbb{F}_{p^2}$ there is some $\varphi(\alpha) \in \mathbb{F}_{p^2}$ such that $(\alpha v)g = \varphi(\alpha)(vg)$. As

$$\varphi(\alpha_1 + \alpha_2)(vg) = ((\alpha_1 + \alpha_2)v)g = (\alpha_1v + \alpha_2v)g = (\varphi(\alpha_1) + \varphi(\alpha_2))(vg),$$

we obtain that φ is additive. Since $\varphi(\alpha\beta)vg = (\alpha\beta v)g = \varphi(\alpha)((\beta v)g) = \varphi(\alpha)\varphi(\beta)(vg)$, we have that φ is a field automorphism. Since φ fixes \mathbb{F}_p pointwise, φ is a member of the Galois group of \mathbb{F}_{p^2} over \mathbb{F}_p . Hence g is a semilinear transformation which gives that $K_6 \leq \Gamma L_2(p^2)$. Therefore $K_6 = \Gamma L_2(p^2)$, as claimed.

To show that K_6 acts irreducibly on U_6 and U_{18} let us take a Singer cycle, i.e., an element g of order $p^4 - 1$ in $\text{GL}_2(p^2) < K_6$. Let $\varepsilon \in \mathbb{F}_{p^4}$ be an eigenvalue of g . Then the eigenvalues of g are $\varepsilon, \varepsilon^p, \varepsilon^{p^2}, \varepsilon^{p^3}$, and so the eigenvalues of $g \wedge g$ on $V \wedge V$ are $\eta = \varepsilon^{1+p}, \eta^p, \eta^{p^2}, \eta^{p^3}, \theta = \varepsilon^{1+p^2}, \theta^p$. The order of η is $(p^4 - 1)/(p + 1) > p^2$, hence $\eta, \eta^p, \eta^{p^2}, \eta^{p^3}$ are all different. Similarly, the order of θ is $p^2 - 1$, so θ and θ^p are distinct. This means that the characteristic polynomial of $g \wedge g$ is the product of two irreducible factors, one of degree 4, the another of degree 2. Hence $V \wedge V$ decomposes into a direct sum of a 4-dimensional and a 2-dimensional irreducible $\langle g \wedge g \rangle$ -submodules. Since U_6 and U_{18} are invariant under $g \wedge g$, we obtain that these are the irreducible summands.

Finally, consider K_{11} . We identify V with the tensor product $\langle u_1, u_2 \rangle \otimes \langle v_1, v_2 \rangle$ by assigning x_1, x_2, x_3, x_4 to $-u_1 \otimes v_1, u_2 \otimes v_1, u_2 \otimes v_2, u_1 \otimes v_2$, respectively. Hence the group $\text{GL}_2(p) \times \text{GL}_2(p)$ acts on V , and the action of the element

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \otimes \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix}$$

is represented by the matrix

$$\begin{pmatrix} \alpha_{11}\beta_{11} & -\alpha_{12}\beta_{11} & -\alpha_{12}\beta_{12} & -\alpha_{11}\beta_{12} \\ -\alpha_{21}\beta_{11} & \alpha_{22}\beta_{11} & \alpha_{22}\beta_{12} & \alpha_{21}\beta_{12} \\ -\alpha_{21}\beta_{21} & \alpha_{22}\beta_{21} & \alpha_{22}\beta_{22} & \alpha_{21}\beta_{22} \\ -\alpha_{11}\beta_{21} & \alpha_{12}\beta_{21} & \alpha_{12}\beta_{22} & \alpha_{11}\beta_{22} \end{pmatrix}. \quad (4)$$

The kernel of the action of $\text{GL}_2(p) \times \text{GL}_2(p)$ on V equals $\{\lambda I \otimes \lambda^{-1} I \mid \lambda \in \mathbb{F}_p^\times\}$. Thus the central product $\text{GL}_2(p) \mathsf{Y} \text{GL}_2(p)$ acts faithfully on V . Let H denote the group of all matrices of the form (4). Elementary, but cumbersome, calculation shows that the group H is the stabilizer of U_{11} , and so $H = K_{11}$. In particular $K_{11} \cong \text{GL}_2(p) \mathsf{Y} \text{GL}_2(p)$. As $\text{GL}_2(p)$ is absolutely irreducible on \mathbb{F}_p^2 , by [Rob96, 8.4.2], K_{11} is irreducible on V .

Consider the subgroups T_1 and T_2 of K_{11} consisting of the elements of the form

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix}.$$

Simple computation shows that the action of a generic element of T_1 on U_{11} is represented by the matrix

$$\begin{pmatrix} \alpha_{11}^2 & -\alpha_{12}^2 & -\alpha_{12}\alpha_{11} \\ -\alpha_{21}^2 & \alpha_{22}^2 & \alpha_{22}\alpha_{21} \\ -2\alpha_{21}\alpha_{11} & 2\alpha_{22}\alpha_{12} & \alpha_{22}\alpha_{11} + \alpha_{12}\alpha_{21} \end{pmatrix}.$$

Since T_1 is isomorphic to $\mathrm{GL}_2(p)$, the derived subgroup $(T_1)'$ is isomorphic to $\mathrm{SL}_2(p)$ and simple computation shows that $(T_1)'$ induces a subgroup of $\mathrm{SL}(U_{11})$. Moreover, $(T_1)'$ preserves the symmetric bilinear form

$$\begin{aligned} &(\alpha_1 x_1 \wedge x_4 + \alpha_2 x_2 \wedge x_3 + \alpha_3(x_2 \wedge x_4 - x_1 \wedge x_3), \beta_1 x_1 \wedge x_4 + \beta_2 x_2 \wedge x_3 + \beta_3(x_2 \wedge x_4 - x_1 \wedge x_3)) \\ &= -\alpha_1 \beta_2 - \alpha_2 \beta_1 - 2\alpha_3 \beta_3. \end{aligned}$$

It is shown in [KL90, Proposition 2.9.1(ii)] that $(T_1)'$ induces $\Omega(\hat{Q})$, where \hat{Q} is the quadratic form induced by the above bilinear form. As $\Omega(\hat{Q})$ is irreducible, we obtain that K_{11} is irreducible on U_{11} . Replacing T_1 by T_2 , the same argument shows that K_{11} is irreducible on U_{11}^\perp . \square

The proof of Theorem 11 is now straightforward.

The proof of Theorem 11. As explained in Lemma 14, G_i is UCS if and only if $\mathrm{GL}(V)_{U_i}$ is irreducible on both V and $\Lambda^2 V/U_i$. Hence Lemmas 13 and 14 imply Theorem 11. \square

UCS p -groups with exponent p^2 are studied in the next section. It follows from Theorems 11 and 6 that a 4-generator exponent- p^2 UCS p -group is a quotient of either $H_{p,4}/N_{16}$ or $H_{p,4}/N_{18}$. Determining precisely which of these two cases leads to UCS p -groups involves subtle isomorphism problems which depend on the value of the prime p . This is illustrated in the the next section.

7. EXTERIOR SELF-QUOTIENT MODULES

The study of UCS p -groups with exponent p^2 is reduced by Theorem 7(b) to considering a problem in representation theory. Recall that the concepts of ESQ-modules and ESQ-groups were defined in Section 3. Unlike the property of irreducibility, the ESQ-property is preserved under subgroups and field extensions, as shown by the following lemma.

Lemma 15. *Let V be an ESQ $\mathbb{F}G$ -module. Then*

- (a) *every subgroup H of G is also an ESQ-subgroup of $\mathrm{GL}(V)$.*

- (b) $V \otimes_{\mathbb{F}} \mathbb{E}$ is an ESQ $\mathbb{E}G$ -module for every extension field \mathbb{E} of \mathbb{F} .
- (c) G contains no non-trivial scalar matrices, and $\dim(V) \geq 3$.

Proof. Parts (a) and (b) are routine to verify. If a scalar matrix λI lies in G , then it follows from $\Lambda^2 V/U \cong V$ that $\lambda^2 = \lambda$ and hence $\lambda = 1$. In addition, $\dim(\Lambda^2 V) \geq \dim(V)$ implies $\dim(V) \geq 3$. Hence part (c) holds. \square

An irreducible ESQ-module can give rise to a smaller dimensional irreducible module over a larger field which does not enjoy the ESQ-property. For example, a 5-dimensional irreducible, but not absolutely irreducible, ESQ-module over \mathbb{F}_q , gives rise to a one dimensional irreducible module over \mathbb{F}_{q^5} which is not an ESQ-module by Lemma 15(c).

Let r and q be coprime integers. Denote the order of q modulo r by $\text{ord}_r(q)$. Then $\text{ord}_r(q)$ is the smallest positive integer n satisfying $q^n \equiv 1 \pmod{r}$. **WARNING:** The variables p , r , and G have different meanings in the following discussion about ESQ-groups, to the previous discussion about UCS-groups.

Theorem 16. *Let p be a prime, q a power of a prime (possibly distinct from p), and let G be a minimal irreducible ESQ-subgroup of $\text{GL}_p(q)$. Then one of the following holds:*

- (a) G is not absolutely irreducible, $r := |G|$ is prime, $\text{ord}_r(q) = p$ and there exist distinct $\alpha, \beta \in \langle q \rangle \leq \mathbb{F}_r^\times$ such that $\alpha + \beta = 1$;
- (b) G is an absolutely irreducible non-abelian simple group;
- (c) G is absolutely irreducible, $|G| = pr^s$ where r is a prime different to p , $\text{ord}_r(q) = 1$, and $s = \text{ord}_p(r)$. Moreover, G' is an elementary abelian group of order r^s and G/G' has order p and acts irreducibly on G' .

Proof. Denote by $V = (\mathbb{F}_q)^p$ the corresponding ESQ $\mathbb{F}_q G$ -module. Here p and $\text{char}(\mathbb{F}_q)$ may be distinct primes. By Lemma 15(a), subgroups of ESQ-groups are ESQ-groups, and hence by the minimality of G , proper subgroups of G act reducibly on V . If H is a non-trivial abelian normal subgroup of G , then by Clifford's theorem either H acts irreducibly on V , or $V = V_0 \oplus V_1 \oplus \cdots \oplus V_{p-1}$ is an internal direct sum of p pairwise non-isomorphic 1-dimensional H -submodules. (If V_0, V_1, \dots, V_{p-1} were all isomorphic, then H would contain non-trivial scalar matrices contrary to Lemma 15(c).)

(a) Suppose that G does not act absolutely irreducibly on V . By Lemma 15(b) we may view G as an ESQ-subgroup of $\text{GL}_p(\mathbb{E})$ where \mathbb{E} denotes the algebraic closure of \mathbb{F}_q . The module \mathbb{E}^p is a direct sum of p pairwise non-isomorphic algebraically conjugate irreducible

1-dimensional G -submodules by [HB82, Theorem VII.1.16]. This proves that G is abelian. We argue that $r := |G|$ is prime. If not, then G has a proper non-trivial subgroup H . By the first paragraph of the proof, V decomposes as the sum of 1-dimensional H -modules. In particular, an element $h \in H$ has an eigenvalue, λ , say, in V . Since h commutes with G , the linear transformation h is a G -endomorphism of V , and so are the transformations λI and $h - \lambda I$. As $h - \lambda I$ is not invertible, Schur's lemma shows that $h - \lambda I = 0$, and so h coincides with the scalar matrix λI . However, as, by Lemma 15(c), G contains no non-trivial scalar matrices, we obtain that $h = I$. Hence the only proper subgroup of G is the trivial subgroup, which shows that r is prime.

Then $\mathbb{E}^p = W \oplus \sigma(W) \oplus \cdots \oplus \sigma^{p-1}(W)$ where $\sigma: \mathbb{E} \rightarrow \mathbb{E}$ is the q th power (Frobenius) automorphism, and $\sigma(W)$ denotes an $\mathbb{E}G$ -module algebraically conjugate to W via σ [HB82, Definition VII.1.13]. Further, $\sigma^p(W) \cong W$. The exterior square of \mathbb{E}^p is isomorphic to a direct sum $\sum_{0 \leq i < j < p} \sigma^i(W) \wedge \sigma^j(W)$. Thus $W \cong \sigma^i(W) \wedge \sigma^j(W)$ for some $0 \leq i < j < p$, as \mathbb{E}^p is an ESQ-module. Suppose that $G = \langle g \rangle$. Then g is conjugate in $\mathrm{GL}_p(\mathbb{E})$ to a diagonal matrix $\mathrm{diag}(\zeta, \zeta^q, \dots, \zeta^{q^{p-1}})$ where $\zeta \in \mathbb{E}^\times$ has order r . It follows from $q^p \equiv 1 \pmod{r}$, that $\mathrm{ord}_r(q)$ equals 1 or p . The first possibility does not arise as g is not a scalar matrix. The condition $W \cong \sigma^i(W) \wedge \sigma^j(W)$ implies that $1 \equiv q^i + q^j \pmod{r}$ where $\alpha = q^i$ and $\beta = q^j$ are distinct powers of q . This completes the proof of part (a).

(b) Suppose now that G is a simple group acting absolutely irreducibly on V . Then G must be non-abelian.

(c) Suppose now that G acts absolutely irreducibly on V and G is not simple. Let N be minimal normal subgroup of G . Then N is a proper subgroup of G and so acts reducibly. Let $V = V_0 \oplus \cdots \oplus V_{p-1}$ be a direct sum of p irreducible 1-dimensional N -submodules. It follows that N is abelian. Suppose that $|N| = r^s$ where r is prime. By the first paragraph of this proof, V_0, \dots, V_{p-1} are pairwise non-isomorphic. Since N has a non-trivial 1-dimensional module over \mathbb{F}_q , it follows that r divides $q - 1$, or $\mathrm{ord}_r(q) = 1$.

By Clifford's theorem, G acts transitively on the set $\{V_0, \dots, V_{p-1}\}$. Choose $g \in G$ that induces a p -cycle. By renumbering if necessary, assume that $V_i g = V_{i+1}$ where the subscripts are read modulo p . Choose $0 \neq e_0 \in V_0$ and set $e_i = e_0 g^i$. Then $e_0 g^p = \lambda e_0$ for some $\lambda \in \mathbb{F}_q$. Since g^p is the scalar matrix λI , it follows from Lemma 15(c) that $\lambda = 1$. Since $\langle g \rangle N$ acts irreducibly on V , it follows by minimality that $G = \langle g \rangle N$ has order pr^s .

We may view G as a subgroup of the wreath product $C_r \wr C_p$. The base group $(C_r)^p$ may be identified with the vector space $(\mathbb{F}_r)^p$, and N may be identified with an irreducible

$\mathbb{F}_r C_p$ -submodule of $(\mathbb{F}_r)^p$. Since the derived subgroup G' equals N , it follows that $r \neq p$. By Maschke's theorem $(\mathbb{F}_r)^p$ is a completely reducible $\mathbb{F}_r C_p$ -module. Since no V_i is the trivial module, N corresponds to an irreducible $\mathbb{F}_r C_p$ -submodule of $(\mathbb{F}_r)^{p-1}$. However, $(\mathbb{F}_r)^{p-1}$ is the direct sum of $(p-1)/\text{ord}_p(r)$ irreducible $\mathbb{F}_r C_p$ -submodules each of dimension $\text{ord}_p(r)$. This proves that $s = \text{ord}_p(r)$, and completes the proof. \square

In Theorem 16(a,c), the order $|N| = r^s$ of a minimal normal subgroup of G is severely restricted. If p and q are given, then r^s must divide $|\text{GL}_p(\mathbb{F}_q)|$, however, if p is given and q is arbitrary, then there are still finitely many choices for r^s .

Theorem 17. *Let $G \leq \text{GL}_p(q)$ be as in Theorem 16(a,c), and let N be a minimal normal subgroup of G . Let $j \in \{2, 3, \dots, p-1\}$, and let $Y_{p,j}$ be the \mathbb{Z} -module*

$$Y_{p,j} = \langle e_0, e_1, \dots, e_{p-1} \mid e_k = e_{1+k} + e_{j+k}, \quad k = 0, 1, \dots, p-1 \rangle$$

where the subscripts are read modulo p . Then the \mathbb{Z} -modules $Y_{p,j}$ are finite, and N is a factor group of $Y_{p,j}$ for some j .

Proof. Cases (a) and (c) can be unified by considering the potentially larger finite field $\mathbb{E} = \mathbb{F}_q(\zeta)$ where ζ has order r . In case (a), the group $G = N$ has order prime r , and $\mathbb{E}^p = V_0 \oplus V_1 \oplus \dots \oplus V_{p-1}$ where V_k is an irreducible 1-dimensional $\mathbb{E}N$ -module corresponding to the eigenvalue $\sigma^k(\zeta)$ where $\sigma \in \text{Gal}(\mathbb{E}/\mathbb{F}_q)$ has order p . Let $g \in \text{GL}_p(\mathbb{E})$ be a matrix of order p satisfying $V_k = V_0 g^k$, for all k . Then $\langle g \rangle N$ is a minimal irreducible ESQ-subgroup of $\text{GL}_p(\mathbb{E})$ with $s = 1$.

Thus we reduce to case (c) where $G = \langle g \rangle N$ is a minimal irreducible ESQ-subgroup of $\text{GL}_p(\mathbb{E})$ where g is as above, $|N| = r^s$ is elementary abelian and $\mathbb{E}^p = V_0 \oplus V_1 \oplus \dots \oplus V_{p-1}$ is a sum of irreducible (but not necessarily algebraically conjugate) 1-dimensional $\mathbb{E}N$ -submodules. As $V_k = V_0 g^k$, each V_k is a non-trivial N -module. If $V_0 \cong V_0 \wedge V_j$, then V_j would be the trivial N -module. Hence we have $V_0 \cong V_i \wedge V_j$ where $0 < i < j < p$.

By replacing g by g^i , we may assume that $V_0 \cong V_1 \wedge V_j$, that is, we may assume that $i = 1$. Suppose henceforth that $V_0 \cong V_1 \wedge V_j$ as N -modules where $1 < j < p$ and $V_k = V_0 g^k$. Suppose $n \in N$ and $n = \text{diag}(\zeta^{x_0(n)}, \zeta^{x_1(n)}, \dots, \zeta^{x_{p-1}(n)})$ where $x_k(n) \in \mathbb{F}_r$ and $\zeta \in \mathbb{E}$ has order r . The N -isomorphisms $V_k \cong V_{1+k} \wedge V_{j+k}$ give rise to equations $x_k(n) = x_{1+k}(n) + x_{j+k}(n)$ in \mathbb{F}_r , where the subscripts are read modulo p . We shall view x_k as an element of the dual space N^* of N . Thus $x_k = x_{1+k} + x_{j+k}$ in N^* for all k .

Our goal now is to relate the abelian group $Y_{p,j}$, which is defined in terms of p and j , to the abelian group N of order r^s . We show now that N^* is an epimorphic image of $Y_{p,j}$. Let

$\mathbb{Z}^p = \langle e_0, \dots, e_{p-1} \rangle$ where $e_k = (0, \dots, 0, 1, 0, \dots, 0)$ has a 1 in position $k + 1$. First, note that $N^* = \langle x_0, x_1, \dots, x_{p-1} \rangle$ because $\cap_{k=0}^{p-1} \ker(x_k)$ is trivial. Second, the homomorphism $\mathbb{Z}^p \rightarrow N^*$ defined by $\sum_{k=0}^{p-1} y_k e_k \mapsto \sum_{k=0}^{p-1} y_k x_k$ is surjective by the previous sentence, and its kernel contains $R_j = \langle e_k - e_{1+k} - e_{j+k} \mid k = 0, 1, \dots, p-1 \rangle$ since $x_k - x_{1+k} - x_{j+k} = 0$ in N^* for all k . As $Y_{p,j} \cong \mathbb{Z}^p / R_j$, this homomorphism induces an epimorphism $Y_{p,j} \rightarrow N^*$. We prove below that $Y_{p,j}$ is finite. Hence $|N^*| = |N| = r^s$ divides $|Y_{p,j}|$.

The above subgroup R_j of \mathbb{Z}^p equals $\mathbb{Z}^p(I - C - C^j)$ where $C = C_p$ denotes the cyclic permutation matrix

$$C_p = \begin{pmatrix} 0 & 1 & & 0 \\ & & \ddots & \\ 0 & 0 & & 1 \\ 1 & 0 & & 0 \end{pmatrix} \in \text{GL}_p(\mathbb{Z}).$$

We argue that $|\mathbb{Z}^p : R_j|$ is finite, or equivalently that $\det(I - C - C^j) \neq 0$. Denote by $\overline{}$ the matrix ring homomorphism $\overline{}: \mathbb{Z}^{p \times p} \rightarrow \mathbb{F}_p^{p \times p}$. Then \overline{C} is conjugate in $\text{GL}_p(\mathbb{F}_p)$ to an upper-triangular matrix with all eigenvalues 1, and $\overline{I - C - C^j}$ is conjugate to an upper-triangular matrix with all eigenvalues -1 . Therefore

$$\det(I - C - C^j) \equiv (-1)^p \pmod{p}.$$

Thus $\det(I - C - C^j) \neq 0$ and $|Y_{p,j}| = |\det(I - C_p - C_p^j)|$ is finite. Alternatively, the formula for the determinant of a circulant matrix shows that

$$\det(I - C_p - C_p^j) = \prod_{k=0}^{p-1} (1 - \xi_p^k - \xi_p^{jk}) \neq 0$$

for $1 < j < p$, where $\xi_p = e^{2\pi i/p}$ denotes a complex primitive p th root of 1. \square

The previous determinant can be defined for any size n . Let $\delta_{n,j} = \det(I - C_n - C_n^j)$ for $1 < j < n$. However, for some composite n , this determinant can be zero. For example, if $n \equiv 0 \pmod{6}$ and $j \equiv -1 \pmod{6}$, then $\delta_{n,j} = 0$ as $1 - \xi_n^{n/6} - \xi_n^{-n/6} = 0$. We can observe that $\delta_{n,n-1} = \pm 1$ whenever $n \equiv \pm 1 \pmod{6}$. To see this note that $(I - C_n - C_n^{-1})^{-1} = I + \sum_{i=1}^{(n-2)/3} (-C_n)^{3i} (I + C_n^{n-1})$ when $n \equiv -1 \pmod{6}$. A similar formula holds for the inverse when $n \equiv 1 \pmod{6}$. Furthermore, one can prove using row operations, and basic properties of Fibonacci numbers that $\delta_{n,2} = 1 + (-1)^n - F_{n-1} - F_{n+1}$. Assume henceforth that $n = p$ is prime.

The structure of the abelian group $Y_{p,j}$ can be determined from the elementary divisors of the Smith normal form of the matrix $I - C_p - C_p^j$. The following table shows, for example, that $Y_{7,3} \cong Y_{7,5} \cong (C_2)^3$ has order 8 and exponent 2, and $Y_{13,12}$ is trivial.

p	3	5	5	7	7	7	11	11	11	11	13	13	13	13	13	13
j	2	2,3	4	2,4	3,5	6	2,6	3,4	5,7,8,9	10	2,7	3,9	4,10	5,8	6,11	12
Divisors	2^2	11		29	2^3		199	67	23		521	131	79	3^3	53	

Table 1: The elementary divisors of $Y_{p,j}$ that are not 1.

Table 1 suggests that $Y_{p,j} \cong Y_{p,k}$ when $jk \equiv 1 \pmod{p}$. Note that there exists a permutation matrix which conjugates C_p to C_p^k , and this conjugates $I - C_p - C_p^j$ to $I - C_p^k - C_p$. Hence $Y_{p,j} \cong Y_{p,k}$.

The following theorem shows that case (b) of Theorem 16 does not arise in dimension 5.

Theorem 18. *Let q be a prime power, and let G be a minimal irreducible ESQ-subgroup of $\text{GL}_5(q)$. Then case (b) of Theorem 16 does not arise, and more can be said about cases (a) and (c):*

- (a) G is not absolutely irreducible, $|G| = 11$, and $\text{ord}_{11}(q) = 5$,
- (c) G is absolutely irreducible of order 55 and $\text{ord}_{11}(q) = 1$.

Furthermore, both of these possibilities occur.

Proof. Suppose that G satisfies case (a) of Theorem 16. Then $r = 11$ by Theorem 17 and Table 1. The subgroup $\langle q \rangle \leq \mathbb{F}_{11}^\times$ has order $p = 5$. Therefore $\langle q \rangle = \{1, 3, 4, 5, 9\} = (\mathbb{F}_{11}^\times)^2$, and $\text{ord}_{11}(q) = 5$ implies that $q \equiv 3, 4, 5, 9 \pmod{11}$. Note that $\alpha = 3$ and $\beta = 9$ satisfy $\alpha + \beta = 1$ in \mathbb{F}_{11} . Conversely, if $\text{ord}_{11}(q) = 5$, then the cyclotomic polynomial $\Phi_{11}(x) = x^{10} + \cdots + x + 1$ factors over \mathbb{F}_q as a product of two distinct irreducible quintics. The companion matrix of either of these quintics generates an irreducible (but not absolutely irreducible) ESQ-subgroup of $\text{GL}_5(\mathbb{F}_q)$ of order 11.

Suppose now that G satisfies case (c) of Theorem 16. As above, $r = 11$. Furthermore, $\text{ord}_{11}(q) = 1$ and $s = \text{ord}_5(11) = 1$. Therefore G is isomorphic to the group $\langle g, n \mid g^5 = n^{11} = 1, g^{-1}ng = n^t \rangle$ of order 55, where $\text{ord}_{11}(t) = 5$. By replacing g by a power of itself, we may assume that $t = 3$. Conversely, there is an irreducible ESQ-subgroup G of $\text{GL}_5(\mathbb{F}_q)$ of order 55 if $q^5 \equiv 1 \pmod{11}$. To see this apply Theorem 21 below with $G = G_L$ where $L = (\mathbb{F}_{11}^\times)^2 = \{1, 3, 4, 5, 9\}$. Note that $\alpha = 3, \beta = 9$ satisfy

$\alpha + \beta = 1$ in \mathbb{F}_{11} , and the condition $q \in L$ is equivalent to $q^5 \equiv 1 \pmod{11}$. Here G_L is a *minimal* ESQ-subgroup if and only if $q \equiv 1 \pmod{11}$.

Suppose now that case (b) of Theorem 16 holds, and G is a non-abelian simple (absolutely) irreducible ESQ-subgroup of $\mathrm{GL}_5(q)$ where $q = p^k$ and $\mathrm{char}(\mathbb{F}_q) = p$. As G is non-abelian simple, we have $G \leq \mathrm{SL}_5(q)$ and $G \cap Z(\mathrm{SL}_5(q)) = 1$. Thus G is isomorphic to an irreducible subgroup of $\mathrm{PSL}_5(q)$. Consider first the case when $p = 2$. The irreducible subgroups of $\mathrm{PSL}_5(2^k)$ were classified by Wagner [Wag78]. As G is simple, it must be isomorphic to one of the following groups: $\mathrm{PSL}_2(11)$, $\mathrm{PSL}_5(2^\ell)$ where $\ell \mid k$, or $\mathrm{PSU}_5(4^\ell)$ where $2\ell \mid k$. Furthermore, each of these possibilities give irreducible subgroups of $\mathrm{PSL}_5(2^k)$. We shall use Lemma 15(a) to show that none of these possibilities give irreducible ESQ-subgroups of $\mathrm{GL}_5(q)$. Since $\mathrm{PSL}_5(2) \leq \mathrm{PSL}_5(2^\ell)$ and $\mathrm{PSL}_2(11) \leq \mathrm{PSU}_5(4^\ell)$ for all ℓ , minimality implies that G is an irreducible (and hence absolutely irreducible) group isomorphic to $\mathrm{PSL}_2(11)$ or $\mathrm{PSL}_5(2)$. The Atlas [WWT⁺08] lists (up to isomorphism) the irreducible 5-dimensional modules for $\mathrm{PSL}_2(11)$ and $\mathrm{PSL}_5(2)$ in characteristic 2. There are four: two for $\mathrm{PSL}_2(11)$ over \mathbb{F}_4 , and two for $\mathrm{PSL}_5(2)$ over \mathbb{F}_2 . Straightforward computation shows that the exterior square of each of these is irreducible. Thus no ESQ-groups arise when $p = 2$.

Suppose now that $p > 2$. Here we use the classification of irreducible subgroups of $\mathrm{PSL}_5(p^k)$ in [DMW79]. To prove that there are no non-abelian simple irreducible ESQ-subgroups of $\mathrm{PSL}_5(p^k)$ (really $\mathrm{GL}_5(p^k)$), it will be convenient by Lemma 15(b) to choose $q = p^k$ to be “sufficiently large.” It follows from [DMW79] that G is isomorphic to one of the following: $\mathrm{PSL}_5(p^\ell)$, $\mathrm{PSU}_5(p^{2\ell})$, $\mathrm{P}\Omega_5(p^\ell)$, $\mathrm{P}\Omega_5(3)$, $\mathrm{PSL}_2(p^\ell)$, A_5 , A_6 , $\mathrm{PSL}_2(11)$, A_7 , M_{11} . Each of these groups contains a subgroup isomorphic to the alternating group A_4 . Indeed, $\mathrm{PSL}_2(p^\ell)$ contains such a subgroup by [Hup67, Satz II.8.18], and we also have

$$A_5 \leq \mathrm{PSL}_2(11) \leq M_{11}, A_5 \leq \mathrm{P}\Omega_5(p) \leq \mathrm{PSL}_5(p^\ell), \mathrm{P}\Omega_5(p) \leq \mathrm{PSU}_5(p^{2\ell}), \text{ and } A_5 \leq \mathrm{P}\Omega_5(3).$$

As A_4 is a subgroup of A_5 , the claim is valid.

Next we show that A_4 does not have a 5-dimensional faithful ESQ-module. Suppose that V is a faithful 5-dimensional A_4 -module over \mathbb{F}_q . Suppose first that the characteristic of \mathbb{F}_q is at least 5, and so the A_4 -modules are completely reducible. If q is large enough, which we may assume by Lemma 15(b), then there are 4 pairwise non-isomorphic irreducible A_4 -modules over \mathbb{F}_q , three of which are 1-dimensional, and one is 3-dimensional. Since A_4 is non-abelian, V decomposes as $V = V_3 + V_1 + V_1'$ where the subscript denotes the dimension. Thus $\Lambda^2 V = \Lambda^2(V_3 + V_1 + V_1')$ contains three 3-dimensional and a 1-dimensional

direct summand, and so it is not ESQ. If the characteristic of \mathbb{F}_q is 3, then there are only two irreducible A_4 -modules, one is 1-dimensional, and the other is 3-dimensional. Thus we may argue the same way as above, except we must use composition factors instead of direct summands. \square

Next we determine the ESQ-subgroups in $\mathrm{GL}_4(\mathbb{F})$ for fields \mathbb{F} of characteristic different from 2. The characteristic 2 case would require additional considerations and it is not relevant to Theorem 1(d). Note that apart from $\mathrm{char}(\mathbb{F}) \neq 2$ we allow \mathbb{F} to be arbitrary, not necessarily a prime field, and it can also have characteristic zero.

Let $L = \mathrm{AGL}_1(5)$ be the group of linear functions of the 5-element field considered as a permutation group of degree 5. We have $L = \langle a, b \rangle$, where $a = (0\ 1\ 2\ 3\ 4)$, $b = (1\ 2\ 4\ 3)$, and $|L| = 20$. Then L naturally embeds into $\mathrm{GL}_5(\mathbb{F})$ for any field \mathbb{F} . If the characteristic of \mathbb{F} is different from 5, then the underlying module splits into a direct sum of submodules $\mathbb{F}^5 = V \oplus V_1$, where $V = \{(x_0, \dots, x_4) \mid x_0 + \dots + x_4 = 0\}$ and $V_1 = \{(x, \dots, x) \mid x \in \mathbb{F}\}$. The action of L on V is absolutely irreducible and it is the only faithful irreducible representation of L over \mathbb{F} . In the following theorem L and V will denote what have just been defined.

Theorem 19. *Let \mathbb{F} be a field of characteristic different from 2 and let $K \leq \mathrm{GL}_4(\mathbb{F})$ be a finite irreducible ESQ-subgroup. Then $\mathrm{char}(\mathbb{F}) \neq 5$, K is isomorphic to a subgroup of L and the action of K is isomorphic to the restriction of the action of L on V . Moreover, 5 divides the order of K , and if 5 is a square in \mathbb{F} then $K \cong L$.*

Proof. First recall [KL90, Prop. 5.5.10] that no finite non-abelian simple group has a non-trivial representation of degree two over a field of characteristic different from 2.

Let M be a minimal normal subgroup of K . We first show that M is abelian. If not, then M is the direct product of pairwise isomorphic non-abelian simple groups. Let S be one of the simple factors. Applying Clifford's Theorem twice for $S \triangleleft M \triangleleft K$, and considering that S has no two-dimensional non-trivial representation, we conclude that S is irreducible. Let $V = (\mathbb{F}_q)^4$. Since $(\Lambda^2 V)/U \cong V$ for some 2-dimensional S -submodule U , the remark in the first paragraph in this proof gives that S acts trivially on U .

Let V^* denote the dual space of V . Let $\psi : \mathrm{Hom}(V, V^*) \rightarrow V \otimes V$ be defined as follows. Let x_1, \dots, x_4 be a basis of V and let x_1^*, \dots, x_4^* be the dual basis of V^* . If $f \in \mathrm{Hom}(V, V^*)$ represented by the matrix $(\alpha_{i,j})$ with respect to these bases, then let $\psi(f)$ be the element $\sum_{i,j} \alpha_{i,j} x_i \otimes x_j$. It is easy to check that ψ is a linear isomorphism.

Now the group S acts on both spaces $\text{Hom}(V, V^*)$ and $V \otimes V$: if $g \in S$, then the matrix of f^g is $g^T(\alpha_{i,j})g$. An easy calculation shows that the isomorphism ψ is an isomorphism of S -modules, and so the fixed points of S in $V \otimes V$ correspond to intertwining operators between the S -modules V and V^* . Identify $\Lambda^2 V$ with $\langle u \otimes v - v \otimes u \mid u, v \in V \rangle \subseteq V \otimes V$. As U is a 2-dimensional subspace of $\Lambda^2 V$ on which S acts trivially, the dimension of these intertwining operators is at least 2. The dimension of the space of these intertwining operators is equal to the dimension of the centralizer algebra of the S -module V . On the other hand, by Schur's lemma, the centralizer algebra is a quadratic extension field \mathbb{E} of \mathbb{F}_q . Further, the S -module V is also an $\mathbb{E}S$ -module. This means that S can be viewed as a subgroup of $\text{GL}_2(\mathbb{E})$, which contradicts the first paragraph of the proof.

So we have that M is an elementary abelian r -group for some prime r , which cannot be the characteristic of \mathbb{F} . Take an extension field $\mathbb{E} \supseteq \mathbb{F}$ containing primitive r th roots of unity and consider $M \leq \text{GL}_4(\mathbb{E})$, which, by Lemma 15(a,b), is an ESQ-group. Now \mathbb{E} is a splitting field for M , so we can fix an eigenbasis $e_1, e_2, e_3, e_4 \in \mathbb{E}^4$ of M .

Suppose that M contains an element without fixed points, i.e., an element $g \in M$ such that 1 is not an eigenvalue of g . Let the eigenvalues of g be $\lambda_i \in \mathbb{E}$ ($1 \leq i \leq 4$). Then the eigenvalues of $g \wedge g$ are $\lambda_j \lambda_k$ ($1 \leq j < k \leq 4$). By the ESQ property there is an injective map $i \mapsto P(i) = \{j, k\}$ such that $\lambda_i = \lambda_j \lambda_k$. Since 1 is not among the eigenvalues of g , we see that $i \notin P(i)$. Up to renumbering the eigenvalues there are only two essentially different injective maps satisfying this property. So we arrive at two alternative systems of equations:

$$\lambda_1 = \lambda_2 \lambda_3, \lambda_2 = \lambda_3 \lambda_4, \lambda_3 = \lambda_4 \lambda_1, \lambda_4 = \lambda_1 \lambda_2; \quad (5)$$

and

$$\lambda_1 = \lambda_2 \lambda_3, \lambda_2 = \lambda_1 \lambda_4, \lambda_3 = \lambda_1 \lambda_2, \lambda_4 = \lambda_1 \lambda_3.$$

It is easy to solve these systems of equations. In the first case we obtain

$$\lambda_1 = \epsilon, \lambda_2 = \epsilon^2, \lambda_3 = \epsilon^4, \lambda_4 = \epsilon^3,$$

where $\epsilon^5 = 1$, and that implies $r = 5$. In the second case the solutions have the form

$$\lambda_1 = \epsilon^2, \lambda_2 = \epsilon^3, \lambda_3 = \epsilon^5, \lambda_4 = \epsilon,$$

where $\epsilon^6 = 1$. However, non-trivial elements of M have prime order r , hence either $\lambda_1 = \epsilon^2 = 1$ or $\lambda_2 = \epsilon^3 = 1$, contrary to our assumption that 1 is not an eigenvalue of g . So the only possibility is that such an element has order 5 and its eigenvalues are all the four distinct primitive fifth roots of unity.

Next we consider the case when 1 is an eigenvalue of every element $g \in M$. We are going to show that this is not possible. Since K is irreducible and $M \triangleleft K$, we have $\{v \in \mathbb{F}^4 \mid \forall g \in M : vg = v\} = 0$. This implies that the trivial module is not a direct summand in the M -module \mathbb{F}^4 , and so it cannot be a direct summand in \mathbb{E}^4 . Hence $\{v \in \mathbb{E}^4 \mid \forall g \in M : vg = v\} = 0$. Now for every fixed i ($1 \leq i \leq 4$) the number of elements $g \in M$ with $e_i g = e_i$ equals $|M|/r$. If an element $g \in M$ has a fixed point in \mathbb{E}^4 (that is, 1 is an eigenvalue of g) then g must fix one of the basis vectors e_i . This shows, for $r > 4$, that the number of elements in M without eigenvalue 1 is at least $|M| - 4|M|/r > 0$, which is not the case now. Hence $r = 2$ or $r = 3$. If M were cyclic, then all non-trivial elements of M would have non-trivial eigenvalues, hence M is non-cyclic in our present case. Then M intersects $\mathrm{SL}_4(\mathbb{F})$ non-trivially, so by the minimality of M we have that all matrices in M have determinant 1.

Let $r = 2$. Let D denote the 8-element subgroup consisting of diagonal matrices (with respect to the basis e_1, e_2, e_3, e_4) with diagonal entries ± 1 and with determinant 1. We have $M \leq D$. By Lemma 15(c), M cannot contain $-I$. Now D has seven maximal subgroups, out of these three contain $-I$ and the remaining four are the stabilizers of the four basis vectors e_i ($1 \leq i \leq 4$) in D . So there remains no possibility for M .

Let $r = 3$, and denote by $\omega \in \mathbb{E}$ a primitive third root of unity. Let $g \in M \leq \mathrm{SL}_4(\mathbb{E})$ and suppose that $g \neq 1$. Then 1 is an eigenvalue of g . If the multiplicity of the eigenvalue 1 is one, then, as $\det g = 1$, the eigenvalues of g are $1, \omega, \omega, \omega$ or $1, \omega^2, \omega^2, \omega^2$. In both cases 1 is not an eigenvalue of $g \wedge g$ contradicting the ESQ property. Hence for every $1 \neq g \in M$ the multiplicity of the eigenvalue 1 is at least two, and then we infer that the eigenvalues of g are $1, 1, \omega, \omega^2$. Now M is a proper subgroup of the group of diagonal matrices with order 3^3 and determinant 1, so it is generated by at most two elements. Since no basis vector can be fixed by both generators, each one of the basis vectors e_1, e_2, e_3, e_4 is fixed by one generator, and the eigenvalue for the other generator on the same eigenvector must be ω or ω^2 . Then the product of the two generators does not have eigenvalue 1, contrary to our assumption.

In summary, we have proved that $r = 5$ and there is a $g \in M$ which has a diagonal matrix $\mathrm{diag}(\epsilon, \epsilon^2, \epsilon^4, \epsilon^3)$ with respect to the basis $e_1, e_2, e_3, e_4 \in \mathbb{E}^4$ (where $\epsilon \in \mathbb{E}$ is a fifth root of unity). Now $g \wedge g$ is $\mathrm{diag}(\epsilon, \epsilon^2, \epsilon^4, \epsilon^3, 1, 1)$ with respect to the basis $e_2 \wedge e_3, e_3 \wedge e_4, e_4 \wedge e_1, e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_4$ of $\Lambda^2 \mathbb{E}^4$. Hence any isomorphism from the $\langle g \rangle$ -module \mathbb{E}^4 into $\Lambda^2 \mathbb{E}^4$ maps e_i to a multiple of $e_{i+1} \wedge e_{i+2}$ (where the indices are taken modulo 4).

Now take an element $h \in \mathbf{C}_K(g) \leq \mathrm{GL}_4(\mathbb{E})$. Since the eigenvalues of g are distinct, h is also diagonal with respect to the basis e_1, e_2, e_3, e_4 , say, $h = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$. The matrix of $h \wedge h$ restricted to $\langle e_2 \wedge e_3, e_3 \wedge e_4, e_4 \wedge e_1, e_1 \wedge e_2 \rangle$ is $\mathrm{diag}(\lambda_2\lambda_3, \lambda_3\lambda_4, \lambda_4\lambda_1, \lambda_1\lambda_2)$. From the ESQ property it follows that the eigenvalues of h satisfy the same system of equations (5) as above, therefore h is a power of g . Thus we have shown that $\langle g \rangle$ is a self-centralizing subgroup of K . In particular, we obtain that $M = \langle g \rangle$ is cyclic of order 5.

Since $M \triangleleft K$ and M is self-centralizing, K is isomorphic to a subgroup in the holomorph of M , which is L . Since the characteristic polynomial of any element g generating M is $x^4 + x^3 + x^2 + x + 1$, $M \leq \mathrm{GL}_4(\mathbb{F})$ is unique up to conjugacy. Hence K can be embedded into a subgroup of $\mathrm{GL}_4(\mathbb{F})$ that is isomorphic to L .

There are three subgroups of L containing M ; namely, M , a dihedral group of order 10, and L . If 5 has a square root in \mathbb{F} , then the dimensions of the irreducible representations of the dihedral group D_5 are at most two, hence proper subgroups of L cannot act irreducibly on \mathbb{F}^4 in this case. \square

Theorem 1(d) is an immediate consequence of the following results.

Theorem 20. *Let p be an odd prime, and let $\{x_1, x_2, x_3, x_4\}$ be a generating set for $H_{p,4}$.*

- (a) *There is no 4-generator UCS 5-group with exponent 5^2 .*
- (b) *If $p \equiv \pm 1 \pmod{5}$, then there is a unique isomorphism class of 4-generator UCS p -groups with exponent p^2 , namely,*

$$\begin{aligned} G_1 = H_{p,4} / \langle & x_1^p[x_1, x_3][x_4, x_1][x_2, x_3]^2[x_4, x_2]^2[x_4, x_3]^2, \\ & x_2^p[x_2, x_1][x_3, x_1]^2[x_4, x_1]^2[x_3, x_2][x_3, x_4]^2, \\ & x_3^p[x_2, x_1]^2[x_1, x_4]^2[x_2, x_3][x_2, x_4]^2[x_3, x_4], \\ & x_4^p[x_1, x_2]^2[x_1, x_3]^2[x_1, x_4][x_3, x_2]^2[x_4, x_2], \\ & [x_1, x_2][x_1, x_4][x_3, x_4], [x_1, x_3][x_2, x_3][x_2, x_4] \rangle. \end{aligned}$$

- (c) *If $p \equiv \pm 2 \pmod{5}$, then there are two isomorphism classes of 4-generator UCS p -groups with exponent p^2 , namely, G_1 as in case (b) and another group G_2 .*

Moreover, $|\mathrm{Aut}(G_1)^{\overline{G_1}}| = 20$ and $|\mathrm{Aut}(G_2)^{\overline{G_2}}| = 5$.

Proof. Let G be a 4-generator UCS group of exponent p^2 , where p is an odd prime. Let K denote $\mathrm{Aut}(G)^{\overline{G}} \leq \mathrm{GL}_4(p)$. By Theorem 7, K is an irreducible ESQ-group. Using Theorem 19 we immediately see that there are no UCS groups of exponent 5^2 with four

generators. Let us use the notation introduced before Theorem 19. In particular, let L denote the group $\text{AGL}_1(5)$ acting on a 4-dimensional vector space V . By quadratic reciprocity, the number 5 is a square in the p -element field if and only if $p \equiv \pm 1 \pmod{5}$. For these primes we have $K = L$, for primes with $p \equiv \pm 2 \pmod{5}$ we can conclude that $K \leq L$ with 5 dividing the order of K .

First assume that $K = L$. Since L has up to equivalence a unique faithful irreducible 4-dimensional representation, we can choose a generating set x_1, x_2, x_3, x_4 of $H = H_{p,4}$ such that the matrices of generators of L acting on \overline{H} will be

$$a = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -1 & -1 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Choosing the basis $[x_1, x_2], [x_1, x_3], [x_1, x_4], [x_2, x_3], [x_2, x_4], [x_3, x_4]$ in H' the action of a and b on H' is described by the matrices

$$a' = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}, \quad b' = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Using the transition matrix

$$t = \begin{pmatrix} 0 & -1 & 1 & -2 & 2 & 2 \\ 1 & 2 & 2 & 1 & 0 & -2 \\ 2 & 0 & -2 & -1 & -2 & -1 \\ -2 & -2 & -1 & 2 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

of determinant $25 \neq 0$ we obtain

$$ta't^{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad tb't^{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

Let W denote the subspace generated by the first four rows of t , and U the subspace generated by the last two rows. Then $H' = W \oplus U$, furthermore V and W are isomorphic L -modules via the isomorphism mapping x_i to the i th row of t for $i = 1, 2, 3, 4$. Since V is an absolutely irreducible L -module, the isomorphism between V and W is determined up to a scalar factor. By Theorem 6(iii) $G = H/N$ for some K -invariant normal subgroup N such that $N \leq \Phi(H)$, $H^p \cap N = 1$ and \overline{H} and $H'/(N \cap H')$ are equivalent K -modules. Therefore, there are exactly $p - 1$ suitable normal subgroups N , but they can be mapped to one another using automorphisms of H sending each x_i to x_i^k for some fixed $k = 1, \dots, p - 1$. Hence there is a unique isomorphism class of those 4-generator UCS groups G of exponent p^2 where the automorphism group of G induces L on \overline{G} . Using the matrices above it is straightforward to write down the defining relations of this group. (In Theorem 20 we avoided inverses by reversing some commutators.)

Now consider the case, when K is a proper subgroup of L . This can happen only if $p \equiv \pm 2 \pmod{5}$. For these primes the five-element cyclic group is an irreducible subgroup in $\text{GL}_4(p)$. Let $E = \text{GF}(p^4)$ and $\epsilon \in E$ a primitive fifth root of unity. Up to conjugation $H = H_{p,4}$ has a unique automorphism a of order 5. Let $M = \langle a \rangle$ and let $W = [H', M]$. Then the 6-dimensional space H' decomposes as a direct sum $H' = W \oplus \mathbf{C}_{H'}(M)$. The M -modules \overline{H} , H^p , and W are isomorphic, and they can be identified with the additive group of E so that the action of a becomes the multiplication by ϵ . Using this identification every M -invariant normal subgroup $N \triangleleft H$ such that H/N is an UCS group with exponent p^2 has the form

$$N_w = \{(\vartheta, w\vartheta) \in H^p \oplus W \mid \vartheta \in E\} \oplus \mathbf{C}_{H'}(M),$$

for $w \in W \setminus \{0\}$. Let γ be a generator of the multiplicative group of E . The multiplication by γ commutes with the multiplication by ϵ , hence W and $\mathbf{C}_{H'}(M)$ are invariant under the action of any automorphism of H that corresponds to the multiplication by γ on \overline{H} . The eigenvalues of the $\text{GF}(p)$ -linear transformation determined by the multiplication by γ on $\overline{H} \cong E$ are $\gamma, \gamma^p, \gamma^{p^2}, \gamma^{p^3}$, and hence its eigenvalues on H' are $\gamma^{p^i + p^j}$ for $0 \leq i < j \leq 3$.

Now γ^{1+p} has degree 4 over $\text{GF}(p)$, so we infer that the eigenvalues on W are $\gamma^{(1+p)p^i}$ for $i = 0, 1, 2, 3$. If $p \equiv 2 \pmod{5}$ then let $k = (1+p)p$, while if $p \equiv 3 \pmod{5}$ then let $k = (1+p)p^2$. In both cases we have $k \equiv 1 \pmod{5}$. Now the multiplication by γ^k on E has the same eigenvalues as the action of γ on W , moreover, $\epsilon^k = \epsilon$, hence the action of γ on W is multiplication by γ^k . Now γ^n transforms N_w to

$$\begin{aligned} & \{(\vartheta\gamma^n, w\vartheta\gamma^{nk}) \in H^p \oplus W \mid \vartheta \in E\} \oplus \mathbf{C}_{H'}(M) \\ &= \{(\vartheta\gamma^n, (w\gamma^{n(k-1)})\vartheta\gamma^n) \in H^p \oplus W \mid \vartheta \in E\} \oplus \mathbf{C}_{H'}(M) = N_{w\gamma^{n(k-1)}}. \end{aligned}$$

A simple calculation using Euclidean algorithm yields that $\gcd(k-1, p^4-1) = 5$. Hence N_{w_1} and N_{w_2} determine isomorphic quotient groups H/N_{w_1} , H/N_{w_2} provided w_1 and w_2 lie in the same coset of the multiplicative group of E modulo the fifth powers. Field automorphisms of E normalize $\langle \gamma \rangle$, hence we can map N_w to

$$\begin{aligned} & \{(\vartheta^p, (w\vartheta)^p) \in H^p \oplus W \mid \vartheta \in E\} \oplus \mathbf{C}_{H'}(M) \\ &= \{(\vartheta^p, w^p\vartheta^p) \in H^p \oplus W \mid \vartheta \in E\} \oplus \mathbf{C}_{H'}(M) = N_{w^p}. \end{aligned}$$

Now it follows that for $G_1 = H/N_1$ the automorphism group induces L on $\overline{G_1}$. Furthermore, the other four cosets modulo the subgroup of fifth powers are permuted cyclically by the Frobenius automorphism, since p is a primitive root modulo 5. Hence for each $w \in E$ which is not a fifth power in E the quotient groups H/N_w are all isomorphic to each other, and this is the group G_2 in Theorem 20(c). \square

It would be possible to write down explicit defining relations of G_2 using a generator element of the multiplicative group of $\text{GF}(p^4)$. However, we thought that a very complicated formula will not help the reader, so we were content with stating the existence of G_2 .

In Theorem 16(a,c) subgroups of the affine general linear group $\text{AGL}_1(\mathbb{F}_{r^s})$ are candidates for minimal irreducible ESQ-groups. We shall clarify when these examples arise, and construct larger ESQ-groups.

Let t be a power of a prime r . Identify the 1-dimensional affine semilinear group $\text{AFL}_1(\mathbb{F}_t)$ with the cartesian product of sets

$$\text{AFL}_1(\mathbb{F}_t) = \text{Gal}(\mathbb{F}_t/\mathbb{F}_r) \times \mathbb{F}_t^\times \times \mathbb{F}_t.$$

Here $\text{Gal}(\mathbb{F}_t/\mathbb{F}_r)$ denotes the group of field automorphisms of \mathbb{F}_t . Multiplication in $\text{AFL}_1(\mathbb{F}_t)$ is defined by

$$(\sigma_1, \lambda_1, \mu_1)(\sigma_2, \lambda_2, \mu_2) = (\sigma_1\sigma_2, (\lambda_1\sigma_2)\lambda_2, (\mu_1\sigma_2)\lambda_2 + \mu_2)$$

where $\sigma_1, \sigma_2 \in \text{Gal}(\mathbb{F}_t/\mathbb{F}_r)$, $\lambda_1, \lambda_2 \in \mathbb{F}_t^\times$ and $\mu_1, \mu_2 \in \mathbb{F}_t$. A 2-transitive action of $\text{A}\Gamma\text{L}_1(\mathbb{F}_t)$ on \mathbb{F}_t is given by

$$\alpha(\sigma, \lambda, \mu) = (\alpha\sigma)\lambda + \mu$$

where $\alpha \in \mathbb{F}_t$, and $(\sigma, \lambda, \mu) \in \text{Gal}(\mathbb{F}_t/\mathbb{F}_r) \times \mathbb{F}_t^\times \times \mathbb{F}_t = \text{A}\Gamma\text{L}_1(\mathbb{F}_t)$.

Theorem 21. *Let t, q be powers of distinct primes. Let $L \leq \mathbb{F}_t^\times$, and suppose that $q \in L$, and there exist distinct $\alpha, \beta \in L$ such that $\alpha + \beta = 1$. Define G_L to be the subgroup of $\text{A}\Gamma\text{L}_1(\mathbb{F}_t)$ containing the elements (σ, λ, μ) such that $\mu \in \mathbb{F}_t$, $\lambda \in L$, and σ induces the identity automorphism $\mathbb{F}_t^\times/L \rightarrow \mathbb{F}_t^\times/L$. Then there exist $|\mathbb{F}_t^\times : L|$ pairwise non-isomorphic absolutely irreducible $\text{ESQ } \mathbb{F}_q G_L$ -modules of dimension $|L|$. In particular, $\text{A}\Gamma\text{L}_1(\mathbb{F}_t)$ is an absolutely irreducible ESQ -subgroup of $\text{GL}_{t-1}(\mathbb{F}_q)$ if $t > 3$ and $\gcd(t, q) = 1$.*

Proof. Let $V = (\mathbb{F}_q)^t$ be the permutation module for $\text{A}\Gamma\text{L}_1(\mathbb{E})$ where $\mathbb{E} := \mathbb{F}_t$ has (prime) characteristic r . Let $(e_\alpha)_{\alpha \in \mathbb{E}}$ be a basis for V indexed by $\alpha \in \mathbb{E}$. The action of $\text{A}\Gamma\text{L}_1(\mathbb{E})$ on V is given by

$$e_\alpha(\sigma, \lambda, \mu) = e_{(\alpha\sigma)\lambda + \mu} \quad (\alpha \in \mathbb{E}).$$

Our proof has two cases. Assume first that $q \equiv 1 \pmod{r}$. In this case the hypothesis $q \in L$ holds trivially, as $1 \in L$ and $q = 1$ in \mathbb{E} . We show later that the second case when $q \not\equiv 1 \pmod{r}$, reduces to this first case.

Let $T: \mathbb{E} \rightarrow \mathbb{F}_r$ denote the absolute trace function: $T(\alpha) = \sum_{\sigma} \alpha\sigma$ where σ ranges over $\text{Gal}(\mathbb{E}/\mathbb{F}_r)$. Let $\zeta \in \mathbb{F}_q^\times$ have order r , and define

$$f_\alpha = \sum_{\mu \in \mathbb{E}} \zeta^{T(\alpha\mu)} e_\mu.$$

Since $|\mathbb{E}|^{-1} \sum_{\alpha \in \mathbb{E}} \zeta^{T(\alpha(\mu-\nu))} = 0$ if $\mu \neq \nu$, and 1 otherwise, we have

$$|\mathbb{E}|^{-1} \sum_{\alpha \in \mathbb{E}} \zeta^{-T(\nu\alpha)} f_\alpha = \sum_{\mu \in \mathbb{E}} \left(|\mathbb{E}|^{-1} \sum_{\alpha \in \mathbb{E}} \zeta^{T(\alpha(\mu-\nu))} \right) e_\mu = e_\nu.$$

Therefore $(f_\alpha)_{\alpha \in \mathbb{E}}$, defines a new basis for V . The action of $\text{A}\Gamma\text{L}_1(\mathbb{E})$ on the new basis is monomial, and given by

$$f_\alpha(\sigma, \lambda, \mu) = \zeta^{-T((\alpha\sigma)\lambda^{-1}\mu)} f_{(\alpha\sigma)\lambda^{-1}\mu}. \quad (6)$$

Consider the normal subgroup $N = \{(1, 1, \mu) \mid \mu \in \mathbb{E}\}$ of G_L . By equation (6), $f_\alpha(1, 1, \mu) = \zeta^{-T(\alpha\mu)} f_\alpha$. Thus each $\langle f_\alpha \rangle$ is an irreducible $\mathbb{F}_q N$ -module. The non-degeneracy

of the map $\mathbb{E} \times \mathbb{E} \rightarrow \mathbb{F}_r: (\alpha, \beta) \mapsto T(\alpha\beta)$ implies that there is an N -module isomorphism

$$\langle f_\alpha \rangle \cong \langle f_\beta \rangle \quad \text{if and only if} \quad \alpha = \beta. \quad (7)$$

Let $\lambda'L$ be a coset of L in \mathbb{E}^\times , and set

$$W(\lambda'L) = \langle f_\alpha \mid \alpha \in \lambda'L \rangle = \sum_{\alpha \in \lambda'L} \langle f_\alpha \rangle.$$

It follows from equation (6) that $W(\lambda'L)$ is a G_L -module. We shall show that it is irreducible. Set $M = \{(1, \lambda, \mu) \mid \lambda \in L, \mu \in \mathbb{E}\}$. Then M is a normal subgroup of G_L , and by (7) the inertia subgroup of $\langle f_\lambda \rangle$ in M is N . Hence by Clifford's theorem, the induced module $\text{Ind}_N^M(\langle f_\lambda \rangle) = W(\lambda'L)$ is M -irreducible, and *a fortiori* G_L -irreducible.

There are two decompositions of V :

$$V = \sum_{\alpha \in \mathbb{E}} \langle f_\alpha \rangle \quad \text{and} \quad V = \langle f_0 \rangle \oplus \sum_{\lambda'L \in \mathbb{E}^\times/L} W(\lambda'L).$$

The first is as a direct sum of irreducible $\mathbb{F}_q N$ -modules, and the second, a direct sum of irreducible $\mathbb{F}_q G_L$ -modules. If $\alpha', \beta' \in \lambda'L$, then by equation (6)

$$(f_{\alpha'} \wedge f_{\beta'}) (\sigma, \lambda, \mu) = \zeta^{-T((\alpha' + \beta')\sigma\lambda^{-1}\mu)} f_{(\alpha'\sigma)\lambda^{-1}} \wedge f_{(\beta'\sigma)\lambda^{-1}}.$$

Hence if $\alpha' \neq \beta'$, then $\langle f_{\alpha'} \wedge f_{\beta'} \rangle \cong \langle f_{\alpha' + \beta'} \rangle$ as $\mathbb{F}_q N$ -modules by (6). Thus $W(\lambda'L)$ is an ESQ-module if and only if $\alpha' + \beta' \in \lambda'L$ for distinct $\alpha', \beta' \in \lambda'L$. Equivalently, $\alpha + \beta = 1$ where $\alpha = \alpha' / (\alpha' + \beta')$ and $\beta = \beta' / (\alpha' + \beta')$ in L are distinct. Clearly $W(\lambda'L) \cong W(\lambda''L)$ as N -modules if and only if $\lambda'L = \lambda''L$. Therefore $W(\lambda'L) \cong W(\lambda''L)$ as G_L -modules if and only if $\lambda'L = \lambda''L$. In summary, the $W(\lambda'L)$ provide $|\mathbb{E}^\times : L|$ pairwise non-isomorphic absolutely irreducible ESQ $\mathbb{F}_q G_L$ -modules of dimension $|L|$.

Suppose now that $q \not\equiv 1 \pmod{r}$, and $q \in L$ holds. We temporarily enlarge the field of scalars from \mathbb{F}_q to the finite field $\mathbb{F}_q(\zeta)$, where ζ has order r . View $W(\lambda'L)$ as an absolutely irreducible ESQ $\mathbb{F}_q(\zeta)G_L$ -module. We shall show that there exists an $\mathbb{F}_q G_L$ -module $U(\lambda'L)$ which satisfies $W(\lambda'L) \cong U(\lambda'L) \otimes_{\mathbb{F}_q} \mathbb{F}_q(\zeta)$. It then follows that the $U(\lambda'L)$ are pairwise non-isomorphic absolutely irreducible ESQ $\mathbb{F}_q G_L$ -modules of dimension $|L|$. By a theorem of Brauer (see [HB82, Theorem VII.1.17] and [GH97]), the module $U(\lambda'L)$ exists if and only if the character χ of $W(\lambda'L)$ has values in \mathbb{F}_q . We shall show that $\chi(\sigma, \lambda, \mu) \in \mathbb{F}_q$ for all $(\sigma, \lambda, \mu) \in G_L$ by showing $\chi(\sigma, \lambda, \mu)^q = \chi(\sigma, \lambda, \mu)$. By equation (6)

$$\chi(\sigma, \lambda, \mu) = \sum_{\{\alpha \in \lambda'L \mid (\alpha\sigma)\lambda^{-1} = \alpha\}} \zeta^{-T((\alpha\sigma)\lambda^{-1}\mu)} = \sum_{\{\alpha \in \lambda'L \mid (\alpha\sigma)\alpha^{-1} = \lambda\}} \zeta^{-T(\alpha\mu)}.$$

However, $\{\alpha \in \lambda' L \mid (\alpha\sigma)\alpha^{-1} = \lambda\} = \{\alpha q \in \lambda' L \mid ((\alpha q)\sigma)(\alpha q)^{-1} = \lambda\}$ as $q \in L$ is fixed by σ . Therefore $\chi(\sigma, \lambda, \mu) = \chi(\sigma, \lambda, \mu)^q$ as desired.

If $L = \mathbb{E}^\times$, then $G_L = \text{A}\Gamma\text{L}_1(\mathbb{E})$. Moreover, $q \in \mathbb{E}^\times$ holds as t, q are powers of distinct primes. If $t > r$, then take α to be an element of \mathbb{E}^\times not in \mathbb{F}_r^\times , and so that $\beta = 1 - \alpha$ satisfies $\alpha + \beta = 1$ and $\alpha \neq \beta$. If $t = r$, then take $\alpha = 2$ and $\beta = -1$. Clearly $\alpha + \beta = 1$ and $\alpha \neq \beta$ provided $r > 3$. This proves that $\text{A}\Gamma\text{L}_1(\mathbb{E})$ is an absolutely irreducible ESQ-subgroup of $\text{GL}_{t-1}(\mathbb{F}_q)$ if $t > 3$ and $\gcd(t, q) = 1$. \square

We next prove Theorem 1(e).

Proof of Theorem 1(e). Consider parts (i) and (iii). Let p be an odd prime, and let $q = p^k$. Let $V = (\mathbb{F}_q)^3$ be the natural $\text{SO}_3(q)$ -module. Choose a basis x_1, x_2, x_3 for V , and the basis $x_2 \wedge x_3, x_3 \wedge x_1, x_1 \wedge x_2$ for $\Lambda^2 V$. The matrix of $g \wedge g$ is $\det(g)(g^{-1})^T$. As $\det(g) = 1$ and $g^T g = I$, it follows that $g \wedge g = g$ and so V is an irreducible ESQ $\text{SO}_3(q)$ -module. By Theorem 7(b) there exists an exponent- p^2 UCS p -group of order q^6 . This proves part (i). Similarly, by the last sentence of Theorem 21, $\text{A}\Gamma\text{L}_1(8)$ is an absolutely irreducible ESQ-subgroup of $\text{GL}_7(\mathbb{F}_q)$ for odd q . Part (iii) now follows by Theorem 7(b).

Consider part (ii). Parts (a) and (b) of Theorem 7 are true with $k = 1$. Thus if G is an exponent- p^2 UCS-group of order p^{10} , then $\text{Aut}(G)^{\overline{G}}$ is an irreducible ESQ-subgroup of $\text{GL}_5(p)$. It follows from Theorem 18 (with $q = p$) that $p^5 \equiv 1 \pmod{11}$. (Additionally, $|\text{Aut}(G)^{\overline{G}}|$ is divisible by 11.) Conversely, if $p^5 \equiv 1 \pmod{11}$, or more generally if $q = p^k$ satisfies $q^5 \equiv 1 \pmod{11}$, then there exists an exponent- p^2 UCS-group of order q^{10} by Theorems 18 and 7. This proves part (ii). \square

Note that $q^{12} = (q^2)^6$ and so by Theorem 1(e)(i) there exist UCS-groups of order q^{12} and exponent p^2 for all powers q of an odd prime p .

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